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MATHEMATICAL MODELING OF WALKING MACHINES WITH ONE-AXIS BODY

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The models walking machine (WM) with one-axis body are considered. Their kinematic schemes are providing the maximum carrying capacity and minimum actuators' power consumption for implementing the specified body movement. The solution of the dynamics equations (DE) for one-axis WM (OWM) are obtained. These DE contain OWM-N kinematic, geometric and simulation parameters, where N is any real number more than 5. The number of mathematical operations obtained in DE are minimal. DE are presented in two forms: first, as a system of differential-algebraic equations where differential equations contain the dynamic reaction at the support points, and algebraic equations describe the relations between the support feet and supporting plane. And secondly, as a system of N second degree differential equations with the excluded relation reactions. Formula of calculating dynamic reactions at the support points is as simple as possible. Formulas for calculating dynamic reactions at pivot points of such WM are derived. The authors also describe algorithms for solving dynamics tasks arising while studying WM walking and give examples.

Keywords: walking machine, plane models, dynamics equations, first task of dynamics, dynamic reactions, moving forces and force moments.

1. Introduction

Finding kinematic schemes and propellers providing the maximum carrying capacity (deadweight) and minimum actuators' power consumption for implementing the specified body movement is an urgent task for WM [1]. Four- and six-legged WM have from 12 to 18 actuators (three actuators on each leg) and provide high kinematic capabilities, maximum smoothness of body movements [1, 2].

If the WM is designed for transporting technological equipment or manipulators, its body does not require to move smoothly. To implement discrete-continuous cycle walk – acceleration, steady motion, deceleration providing a predetermined body movement – it does not need to have four or six universal legs and several motion freedoms in the three-point state. It is enough to have, for example, two legs, each of which has two feet and one actuator, and one support leg (crutch) with one foot and low-power actuators [3, 4]. The kinematic capabilities of such WM are minimum. The WM like this has one motion freedom while making a three-point step. We should expect that the minimum number of legs and actuators of electromechanical WM, as well as the efficient walking control with power recuperation of actuators in the deceleration cycle will provide a high specific carrying capacity (WM deadweight ratio) and low specific power consumption. This paper continues the investigation described in articles [2, 4] and contains the research results for such WMs.

The paper [2] shows WMs modeled with some linkwork (L) on a plane and proposes kinematic analysis models of the walking process for a three-point WM with one, two and three motion freedoms. The paper [4] proposes walking modeling and animation algorithms for such WMs. Specific examples show techniques of WM visualization in statics and dynamics, approaches for manual and automatic WM motion animation and describe simulation environment for manual walking control system. The use of open distributed information technologies (SVG, CAB, JavaScript) for these purposes is also described. MVC architecture is applied to build a simulation environment for manual walking control.

From the WMs considered in the papers [2, 4] we select WM with one body motion freedom. We identify such WM as one-axis_and denote them OWM. OWM class can be divided into subclasses depending on the total number of units N. We denote the specific subclass as OWM-N. For example, Fig. 1 show schemes of OWM-5, Fig. 2 – schemes OWM-6, Fig. 3 – schemes OWM-7, Fig. 4 – schemes of OWM-8. These OWM hinges axes on the supporting plane (SP) are directed against the gravitation, i.e. the actuators do not overcome these forces. If the feet mechanisms of such OWM have an irreversible gearbox and provide their reciprocating motion along the axes that are collinear to the actuators' hinges axes, then the feet actuators do not work against the gravitation forces in the reference position. These properties of the OWM schemes and the opportunity of using only one actuator (e.g., OWM-5 front axle actuator in Fig. 1a) provide low power consumption.

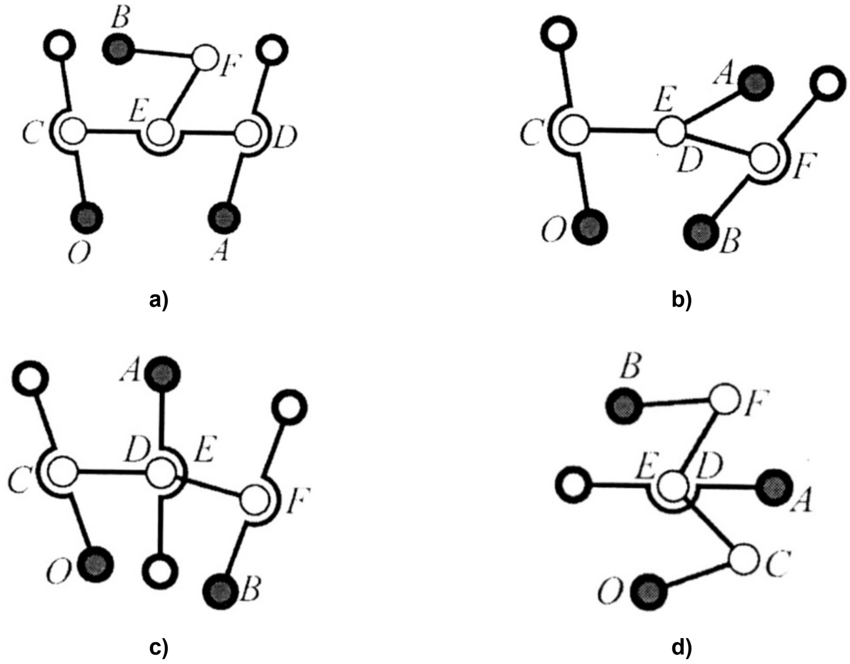


Fig. 1. OWA with 5 units

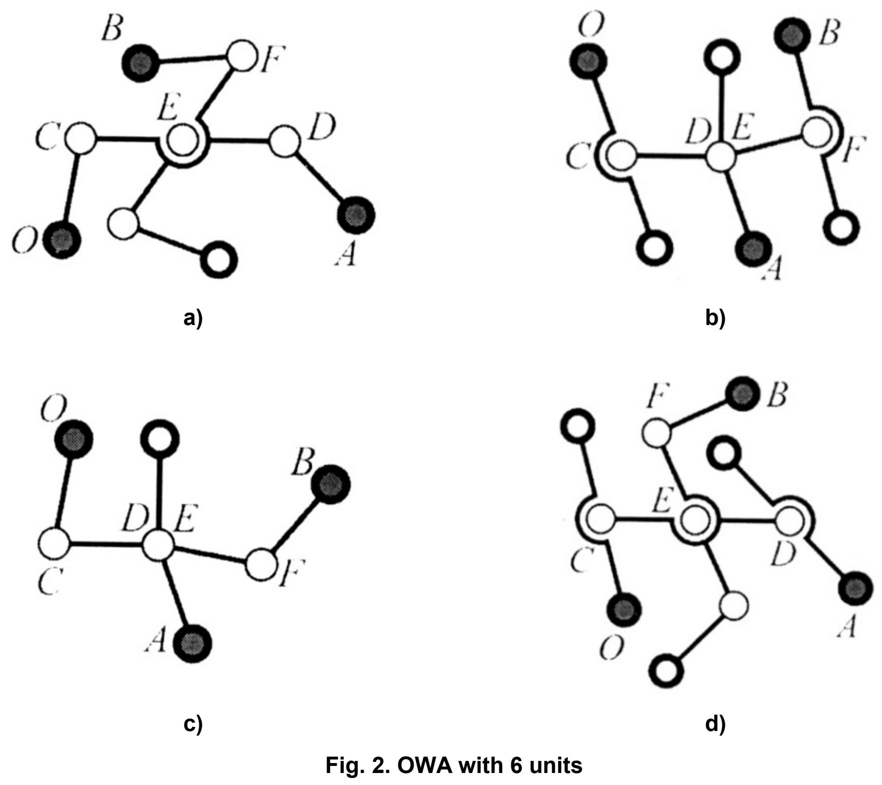
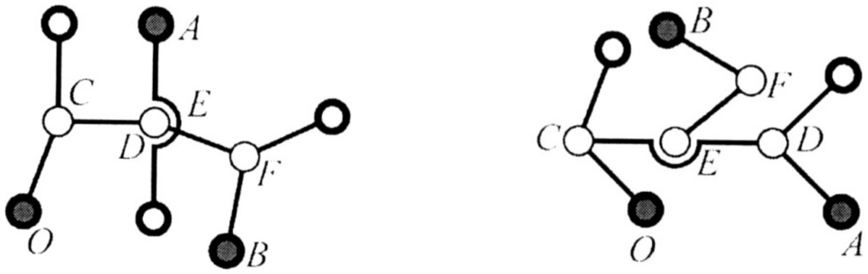
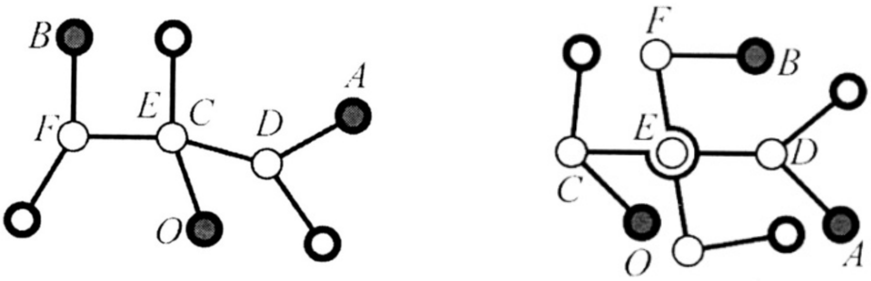


Fig. 2. OWA with 6 units



a) b)

Fig. 3. OWA with 7 units



a) b)

Fig. 4. OWA with 8 units

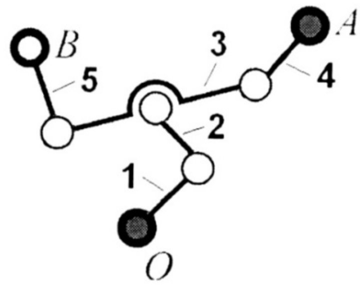


Fig. 5. OWA with contour units

The solution of the dynamics equations (DE) for OWM is published in the present paper. These DE must contain OWM-N kinematic, geometric and simulation parameters, where N is any real number more than 5. The number of mathematical operations obtained in DE shall be minimal. DE shall be presented in two forms: first, as a system of differential-algebraic equations where differential equations contain the dynamic reaction at the support points, and algebraic equations describe the relations between the support feet and SP. And secondly, as a system of N second degree differential equations with the excluded relation reactions. Formula of calculating dynamic reactions at the support points shall be as simple as possible.

The result of these equations was done with new methods of deriving bodies system DE described in the papers [3, 5–7]. It made possible to obtain DE with explicit options in a simple form that did not allow further simplification of these equations. The subsystems masses, static and inertia moments of complemented bodies introduced in the papers [3, 5–7] are constant inertial parameters.

In fact, DE and other design formulas proposed in this paper, apply to a larger set of L than described in the paper title. There are many L of this set with contour units (elements) that form the structure shown in Fig. 5, where each contour unit can carry any tree-type L, i.e. every contour unit can be a base for L with open branches. The contour units with serial number 3 is a body which has suspended single-unit legs (Fig. 5 shows only supporting legs 4 and 5) and multi-unit legs (Fig. 5 shows only one supporting two-unit leg). The hinges at points O, A and B connecting the SP with the end units (shins) of legs model support feet. The other legs and their units are not shown in Fig. 5. But if any, they are transferred to a new (target) support position on the SP. And portable legs can be suspended not only to the body (unit 3), but also to any of the elements with serial number 1, 2, 4, 5. While walking, in each position, the contour units must have the structure and numbers as in Fig. 5. For this purpose, the elements are renumbered in a new state of OWM.

We introduce the following notation for the moments of driving forces generated by the actuators in the hinges of the leg units: M_b is the moment of force about the pivot pin [2] connecting the body (unit 3 in Fig. 5) and the thigh (unit 2 in Fig. 5) of a two-unit support leg; M_g is the moment of force about the pivot pin connecting the thigh and the shin (unit 1 in Fig. 5) of a two-unit support leg; M_4 is the moment of force about the pivot pin connecting the body and the shin of the supporting single-unit leg numbered 4; M_5 is the moment of force about the pivot pin connecting the body and the shin of the supporting single-unit leg numbered 5. If $N > 5$ (N is the number of OWM units), then M_i is the moment of force about the pivot pin connecting the removable unit numbered i with the preceding unit (towards the body).

The common features for all OWM are the following. First, they have five contour units (elements). Secondly, two contour units (units 4 and 5), forming the hinges with the SP, have one common base (body). These hinges simulate the supporting OWM feet at points A and B. A three-point OWM can be considered as L with two branches closed at points A and B [3]. The first unit (in order) is the element forming the hinge with SP at the support point O. The other units (if any) are portable.

2. OWM DE with relations at support points A and B

We denote as j.i a formula or a statement (i) in the paper (j) from the references.

Statement 1. OWM DE can be represented in the following vector-matrix form,

$$H \cdot \ddot{\alpha} + h \cdot \dot{\alpha}^2 - G - S \cdot M = M_r, \qquad (1)$$

where the upper left blocks of 5×5 matrices H, h and S are represented as

$$H_o = \begin{pmatrix} J_1 & H_{21} & H_{31} & H_{41} & H_{51} \\ H_{21} & J_2 & H_{32} & H_{42} & H_{52} \\ H_{31} & H_{32} & J_3 & H_{43} & H_{53} \\ H_{41} & H_{42} & H_{43} & J_4 & 0 \\ H_{51} & H_{52} & H_{53} & 0 & J_5 \end{pmatrix}, \quad h_o = \begin{pmatrix} 0 & -h_{21} & -h_{31} & -h_{41} & -h_{51} \\ h_{21} & 0 & -h_{32} & -h_{42} & -h_{52} \\ h_{31} & h_{32} & 0 & -h_{43} & -h_{53} \\ h_{41} & h_{42} & h_{43} & 0 & 0 \\ h_{51} & h_{52} & h_{53} & 0 & 0 \end{pmatrix},$$

$$S_o = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The column vector of the moments of driving forces in the hinges is $M = (0, -M_g, -M_b, M_4, M_5, ...)^T$, where the driving moments of the portable leg units (if any) are in place of dots of N-dimensional column vector M, i.e. M_6, M_7, M_8, if $N > 5$. Non-zero elements of the column vector $M_r = (M_{r1}, M_{r2}, M_{r3}, M_{r4}, M_{r5}, ...)^T$ are calculated by the following formulas

$$M_{r1} = R_2(y_a + y_b)c_{\beta 1} - R_2(x_a + x_b)s_{\beta 1}, \quad M_{r2} = R_3(y_a + y_b)c_{\beta 2} - R_3(x_a + x_b)s_{\beta 2},$$

$$M_{r3} = R_4(y_a c_{\beta 3} - x_a s_{\beta 3}) + R_5(y_b c_{\gamma 3} - x_b s_{\gamma 3}), \quad M_{r4} = R_a(y_a c_{\beta 4} - x_a s_{\beta 4}),$$

$$M_{r5} = R_b(y_b c_{\beta 5} - x_b s_{\beta 5}).$$

If a three-point OWM has portable units, i.e. if $N > 5$, there are zeros in place of dots of the N-dimensional column vector. The following geometrical relations are imposed on the rotation angles of the contour units

$$\begin{cases} R_2 c_{\beta 1} + R_3 c_{\beta 2} + R_4 c_{\beta 3} + R_a c_{\beta 4} = A_x, \quad R_2 s_{\beta 1} + R_3 s_{\beta 2} + R_4 s_{\beta 3} + R_a s_{\beta 4} = A_y, \\ R_2 c_{\beta 1} + R_3 c_{\beta 2} + R_5 c_{\gamma 3} + R_b c_{\beta 5} = B_x, \quad R_2 s_{\beta 1} + R_3 s_{\beta 2} + R_5 s_{\gamma 3} + R_b s_{\beta 5} = B_y, \end{cases} \qquad (2)$$

where R_a is the distance from the pivot pin of the fourth unit to the reference point A; R_b is the distance from the pivot pin of the fifth unit to the reference point B; A_x, A_y are the coordinates of the reference point A in the coordinate system (CS) O**ij**, rigidly connected with SP; B_x, B_y are the coordinates of reference points B in the CS O**ij**; x_a, y_a are the projections of reaction force at the reference point A on the axis O**i**, O**j**; x_b, y_b are the projections of the reaction force at the reference point B on the axis O**i**, O**j**; $s_{\beta i} = \sin(\beta_i + \alpha_i)$, $c_{\beta i} = \cos(\beta_i + \alpha_i)$, α_i is the angle from the axis O**i** to the axis $O_i\mathbf{i_i}$ directed to mass center of the i-th augmented WM unit [3]; β_i is the angle from axis $O_i\mathbf{i_i}$ to axis $O_i\mathbf{e_{i+1}}$ for i = 1, 2, 3 ($\mathbf{e_i} = \mathbf{O_{i-1}O_i}/R_i$), β_4 is the angle from axis $O_4\mathbf{i_4}$ to axis $\mathbf{O_4A}$, β_5 is the angle from axis $O_5\mathbf{i_5}$ to axis $\mathbf{O_5B}$; $s_{\gamma 3} = \sin(\gamma + \alpha_3)$, $c_{\gamma 3} = \cos(\gamma + \alpha_3)$, γ is the angle from axis $O_3\mathbf{i_3}$ to axis $O_3\mathbf{e_5}$. The other notations and values are described in statement 3.7, in particular, the H_{ki} elements of matrix $H = \{H_{ki}\}_{N\times N}$ are calculated according to the formula (3.21), the elements h_{ki} of matrix $h = \{h_{ki}\}_{N\times N}$ are calculated according to the formula (3.22), the elements S_{ki} of matrix $S = \{S_{ki}\}_{N\times N}$ are calculated according to the formula (3.23), $G = (G_1, G_2, G_3, ..., G_N)^T$.

Proof. We can use the following algorithm to write L DE [3]:

1. We select one of the units forming a hinge with the SP as the first unit in order. We select the shin of a two-unit leg.
2. All units are numbered 2, 3, ..., N consistently. We use the numbers as in Fig. 5.
3. We break mentally the relations (hinges) at the support points A and B and replace them with the reaction forces $\mathbf{F_{r4}} = \mathbf{F_a}$ and $\mathbf{F_{r5}} = \mathbf{F_b}$ where 4 and 5 are the numbers of units that form broken relations with SP.

According to (3.21), (3.22), (3.23) we get the blocks H_o, h_o, S_o of matrices H, h, S for contour OWM units. Really, the only zero element located under diagonal to blocks H_o and h_o is situated at the intersection of the 5th row and 4th column, as $4 \not\in \{1...5\}$ [3]. In the matrix S the element $S_{45} = 0$ because the base of the 5th level is the third unit, not the fourth. The elements $S_{34} = S_{35} = -1$, as the 3rd unit is the base for the 4th and 5th units, etc. 1

The element M of N-dimensional column vector M_1=0 since the first contour level (see Fig. 5) forms a hinge with the SP. The driving forces of the actuator don't work in this hinge. The element $M_2 = -M_g$ since M_g is the moment applied to the shin of a two-unit leg about a thigh of this leg, and according to the definition [3] M_2 is the moment applied to the 2nd unit of the L (to the thigh of a two-unit OWM leg) about 1-st unit of L (shin of a two-unit OWM leg). The element $M_3 = -M_b$ since M_b is the moment applied to the thigh of a two-unit leg about the OWM body, and according to the definition [3] M_3 is the moment applied to the 3rd unit of L (to the OWM body) about the 2nd unit of L (OWM thigh).

By the definition [3] M_{rk} is the moment of force acting on the k-th unit, and is caused by the reaction forces of broken connections and $M_{rk} = L_k\mathbf{k}\cdot\mathbf{p_k}\times\mathbf{F_{rk}} + \sum_{i,k} R_i\mathbf{k}\cdot\mathbf{e_i}\times\sum_{j\ge i}\mathbf{F_{rj}}$, where $\bar{k}$ is the unit vector of normal to AB; $\mathbf{F_{rk}}$ is the reaction forces applied to the k-th unit (if the k-th unit is not closed on SP, then $\mathbf{F_{rk}} = 0$); L_k is the distance from the pivot point of the k-th unit to the point of force application $\mathbf{F_{rk}}$; $\mathbf{p_k}$ is an ort directed from point O_k to the point of force application $\mathbf{F_{rk}} = x_{rk}\mathbf{i} + y_{rk}\mathbf{j}$ according to (3.25). According to $\mathbf{e_i} = c_{\beta i-1}\mathbf{i} + s_{\beta i-1}\mathbf{j}$, $\mathbf{p_k} = c_{\beta k}\mathbf{i} + s_{\beta k}\mathbf{j}$ and $\mathbf{k}\cdot\mathbf{p_k}\times\mathbf{F_{rk}} = \mathbf{k}\times\mathbf{p_k}\cdot\mathbf{F_{rk}}$, $\mathbf{k}\times\mathbf{p_k} = c_{\beta k}\mathbf{j} - s_{\beta k}\mathbf{i}$, $\mathbf{k}\times\mathbf{e_i} = c_{\beta i-1}\mathbf{j} - s_{\beta i-1}\mathbf{i}$, we get

$$M_{rk} = L_k(c_{\beta k}\mathbf{j} - s_{\beta k}\mathbf{i})\cdot(x_{rk}\mathbf{i} + y_{rk}\mathbf{j}) + \sum_{i,k} R_i \sum_{j\ge i}(c_{\beta i-1}\mathbf{j} - s_{\beta i-1}\mathbf{i})\cdot(x_{rj}\mathbf{i} + y_{rj}\mathbf{j}) =$$

$$= L_k(y_{rk}c_{\beta k} - x_{rk}s_{\beta k}) + \sum_{i,k} R_i \sum_{j\ge i}(c_{\beta i-1}y_{rj} - s_{\beta i-1}x_{rj}).$$

For OWM $\mathbf{F_{ri}} = 0$ (i = 1, 2, 3), $\mathbf{F_{r4}} = \mathbf{F_a}$, $\mathbf{F_{r5}} = \mathbf{F_b}$, $L_4 = R_a$, $L_5 = R_b$, $x_{r4} = x_a$, $y_{r4} = y_a$, $x_{r5} = x_b$, $y_{r5} = y_b$. Hence

$$M_{r1} = R_2(c_{\beta 1}y_a - s_{\beta 1}x_a + c_{\beta 1}y_b - s_{\beta 1}x_b),\quad M_{r2} = R_3(c_{\beta 2}y_a - s_{\beta 2}x_a + c_{\beta 2}y_b - s_{\beta 2}x_b),$$

$$M_{r3} = R_4(c_{\beta 3}y_a - s_{\beta 3}x_a) + R_5(c_{\gamma 3}y_b - s_{\gamma 3}x_b),\quad M_{r4} = R_a(y_a c_{\beta 4} - x_a s_{\beta 4}),\quad M_{r5} = R_b(y_b c_{\beta 5} - x_b s_{\beta 5}).$$

Bond equations (2) are a coordinate form of demonstrable vector equations

$\mathbf{OO_2} + \mathbf{O_2O_3} + \mathbf{O_3O_4} + \mathbf{O_4A} = \mathbf{OA}$, $\mathbf{OO_2} + \mathbf{O_2O_3} + \mathbf{O_3O_5} + \mathbf{O_5B} = \mathbf{OB}$.

The statement is proved.

Please pay attention to the following observations before using statement 1.

Note 1. Statement 1 allows to record DE of OWM in the form containing the relation response only in two reference points A and B. The reference point O is represented by a hinge connecting the shin of a two-unit leg with SP. In this case the number of DE and geometric relations equations are minimal. The analytical work is simplified for the relationships minimization. If there is no problem of calculating responses in the reference point O, for example, if OWM moves on a plane and a two-unit leg is the supporting one (crutch) [3], i.e. the vector of dynamic response at the point O is perpendicular to SP, so this approach is optimal as a minimization of computational work.

Note 2. Another approach to derive DE of OWM has a relation with the corollary 6.3 which presents the L DE on the free base. OWM body is considered to be free. The number of DE will increase by two and they will contain the dynamic response at three points (O, A and B). The number of geometric relations will also increase by two. The formulas to calculate the dynamic responses in all control points can be obtained from these equations. Then they can be used to exclude reactions from OWM DE. Both approaches lead to the same form of OWM DE with the excluded relations reactions. This fact can be used to check and control the proper formulation and numerical experiments.

Note 3. The number of equations in (1), (2) is N + 4. Hence, we get N rotation angles of the units $\alpha_i(t)$ (i = 1, ..., N) and four projections $x_a(t)$, $y_a(t)$, $x_b(t)$, $y_b(t)$ of reactions at the support points A and B from the system of equations (1), (2) in a given initial state of OWM, for example, with the given values of elements of N-dimensional column vectors $\alpha(0)$, $\dot{\alpha}(0)$ and with the given laws of changing the moments of driving forces in the hinges of the leg. To do this, first we can use bond equations (2) as the first given integrals of the differential equations (1). These integrals allow to exclude from (1) four of the required rotation angle of contour units and their derivatives. Then we can deduce x_a, y_a, x_b, y_b from the first four equations of the system (1) and substitute them in the fifth equation. This equation together with the others (if N > 5) will not contain dynamic reactions and can be used to calculate rotation angles of the portable units (if N > 5) and the angle of rotation of one contour unit. First we can use the system (1) to exclude bond reactions. We will do it in the next section, i.e. we will use DE (1) for deriving formulas of calculating the dynamics reactions at the support points A, B and the showing the OWM DE with excluded bond reactions.

Example 1. We denote OWM-5 in Fig. 1 as OWM-5a. We write DE (1) and relation equations (2) for OWM-5a. In this and the following examples we assume that the mass centers of single-unit legs are on the axes of their rotation, the lengths of these legs are equal ($R_a = R_b = R$) for this OWM-5a. The body mass center is at the point of suspension of a two-unit leg. The axis $O_3\mathbf{i_3}$ is directed to the point O_4. $R_4 = R_5 = L$ is equal a half of the body length, $R_2 = a$ is the shin length, $R_3 = b$ is the thigh length of a two-unit leg. The thigh mass center is on axis O_2O_3. The shin mass center is on axis O_1O_2. It is true that $\beta_i = 0$ for all i, $\gamma = \pi$ and $d_3 = d_4 = d_5 = 0$, $G_3 = G_4 = G_5 = 0$ for this OWM-5a. Non-zero elements of the matrices H and h according to (3.21), (3.22) are calculated by the formulas $H_{ki} = m_k d_k R_{i+1} \cos(\mathbf{e_{i+1}}, \mathbf{i_k})$, $h_{ki} = m_k d_k R_{i+1} \sin(\mathbf{e_{i+1}}, \mathbf{i_k})$. Therefore, $H_{3i} = H_{4i} = H_{5i} = 0$ and $h_{3i} = h_{4i} = h_{5i} = 0$. Under the definition $(\mathbf{e_2}, \mathbf{i_2}) = \alpha_2 - \alpha_1$, i.e. $H_{21} = d \cdot \cos(\alpha_2 - \alpha_1)$, $h_{21} = d \cdot \sin(\alpha_2 - \alpha_1)$ where $d = m_2 d_2 R_2 = m_2 d_2 a$. We substitute the values in the matrices and the column vectors of DE (1) and perform matrix operations. Then we get the desired system of five nonlinear differential equations of 2nd degree in the following form

$$\begin{cases} J_1\ddot{\alpha}_1 + H_{21}\ddot{\alpha}_2 - h_{21}\dot{\alpha}_2^2 - G_1 - M_g = a(y_a + y_b)\cos\alpha_1 - a(x_a + x_b)\sin\alpha_1, \\ H_{21}\ddot{\alpha}_1 + J_2\ddot{\alpha}_2 + h_{21}\dot{\alpha}_1^2 - G_2 + M_g - M_b = b(y_a + y_b)\cos\alpha_2 - b(x_a + x_b)\sin\alpha_2, \\ J_3\ddot{\alpha}_3 + M_b + M_4 + M_5 = L(y_a - y_b)\cos\alpha_3 - L(x_a - x_b)\sin\alpha_3, \\ J_4\ddot{\alpha}_4 - M_4 = R(y_a\cos\alpha_4 - x_a\sin\alpha_4), \quad J_5\ddot{\alpha}_5 - M_5 = R(y_b\cos\alpha_5 - x_b\sin\alpha_5). \end{cases}$$

It is obviously that relation equations (2) in the accepted notation for OWM-5a have the form

$$\begin{cases} a\cos\alpha_1 + b\cos\alpha_2 + L\cos\alpha_3 + R\cos\alpha_4 = A_x, \quad a\sin\alpha_1 + b\sin\alpha_2 + L\sin\alpha_3 + R\sin\alpha_4 = A_y, \\ a\cos\alpha_1 + b\cos\alpha_2 - L\cos\alpha_3 + R\cos\alpha_5 = B_x, \quad a\sin\alpha_1 + b\sin\alpha_2 - L\sin\alpha_3 + R\sin\alpha_5 = B_y. \end{cases}$$

3. Formula of calculating dynamic reactions at the support points

For walking it is necessary for the dynamic reactions at the support points to get into the friction cones [1]. Otherwise, the feet in the support points will be out of the SP. The contact with the friction cones can be achieved due to proper distribution of driving force moments at hinges of the leg units.

Statement 2. For OWM the projection of force reactions at the support points A and B are calculated by the formulas

$$x_a = (c_{\beta4}D + c_{\beta5}D_4/R_a)/s_{\beta54}, \quad y_a = (s_{\beta4}D + s_{\beta5}D_4/R_a)/s_{\beta54}, \tag{3}$$

$$x_b = (c_{\beta2}D_1/R_2 - c_{\beta1}D_2/R_3)/s_{\beta21} - x_a, \quad y_b = (s_{\beta2}D_1/R_2 - s_{\beta1}D_2/R_3)/s_{\beta21} - y_a, \tag{4}$$

where D_i is the left part of the i-th equation of system (1),

$$D = (s_{\beta52}D_1/R_2 - s_{\beta51}D_2/R_3)/s_{\beta21} + D_5/R_b, \quad s_{\beta ji} = \sin(\beta_j + \alpha_j - \beta_i - \alpha_i).$$

Proof. The proof contains an algorithm for deducing formulas of calculating the desired reactions. This algorithm is based on solving a system of 4 linear algebraic equations that are the 1st, 2nd, 4th and 5th of DE (1) with quantities x_a, y_a, x_b, y_b. From the first two equations of system (1) we will get the following system of linear algebraic equations and its determinant Δ_1 to calculate $x_a + x_b$, $y_a + y_b$

$$\begin{cases} -s_{\beta1}(x_a + x_b) + c_{\beta1}(y_a + y_b) = D_1/R_2, \\ -s_{\beta2}(x_a + x_b) + c_{\beta2}(y_a + y_b) = D_2/R_3, \end{cases} \quad \Delta_1 = \begin{vmatrix} -s_{\beta1} & c_{\beta1} \\ -s_{\beta2} & c_{\beta2} \end{vmatrix} = s_{\beta2}c_{\beta1} - c_{\beta2}s_{\beta1}.$$

By using $s_{\beta21} = \Delta_1 = \sin(\beta_2 + \alpha_2 - \beta_1 - \alpha_1)$ and the formula of sine of two numbers difference, we get

$$x_a + x_b = \begin{vmatrix} D_1/R_2 & c_{\beta1} \\ D_2/R_3 & c_{\beta2} \end{vmatrix} / \Delta_1 = (c_{\beta2}D_1/R_2 - c_{\beta1}D_2/R_3)/s_{\beta21},$$

$$y_a + y_b = \begin{vmatrix} -s_{\beta1} & D_1/R_2 \\ -s_{\beta2} & D_2/R_3 \end{vmatrix} / \Delta_1 = (s_{\beta2}D_1/R_2 - s_{\beta1}D_2/R_3)/s_{\beta21}.$$

By using $D_x = (c_{\beta2}D_1/R_2 - c_{\beta1}D_2/R_3)/s_{\beta21}$, $D_y = (s_{\beta2}D_1/R_2 - s_{\beta1}D_2/R_3)/s_{\beta21}$ we get $x_a + x_b = D_x$, $y_a + y_b = D_y$.

From the fifth equation of system (1) will get $y_b c_{\beta5} - x_b s_{\beta5} = D_5/R_b$. We put expressions $x_b = D_x - x_a$, $y_b = D_y - y_a$ instead of x_b, y_b. Then we get the following equation

$$x_a s_{\beta5} - y_a c_{\beta5} = D_x s_{\beta5} - D_y c_{\beta5} + D_5/R_b. \tag{5}$$

Taking into account D_x, D_y and $s_{\beta ji}$ the right part of equation (5) can be represented in the form

$$D = [(c_{\beta2}D_1/R_2 - c_{\beta1}D_2/R_3)s_{\beta5} - (s_{\beta2}D_1/R_2 - s_{\beta1}D_2/R_3)c_{\beta5}]/s_{\beta21} + D_5/R_b =$$

$$= [D_1(s_{\beta5}c_{\beta2} - c_{\beta5}s_{\beta2})/R_2 - D_2(s_{\beta5}c_{\beta1} - c_{\beta5}s_{\beta1})/R_3]/s_{\beta21} + D_5/R_b =$$

$$= (D_1 s_{\beta52}/R_2 - D_2 s_{\beta51}/R_3)/s_{\beta21} + D_5/R_b. \tag{6}$$

By using the fourth equation of system (1) we get the following system of linear algebraic equations to calculate x_a, y_a

$$\begin{cases} -s_{\beta4}x_a + c_{\beta4}y_a = D_4/R_a, \\ s_{\beta5}x_a - c_{\beta5}y_a = D. \end{cases} \tag{7}$$

The determinant of this system is $\Delta_2 = s_{\beta4}c_{\beta5} - c_{\beta4}s_{\beta5} = -s_{\beta54}$. So

$$x_a = \begin{vmatrix} D_4/R_a & c_{\beta4} \\ D & -c_{\beta5} \end{vmatrix} / (-s_{\beta54}) = \frac{c_{\beta4}D + c_{\beta5}D_4/R_a}{s_{\beta54}},$$

$$y_a = \begin{vmatrix} -s_{\beta 4} & D_4/R_a \\ s_{\beta 5} & D \end{vmatrix} /(-s_{\beta 54}) = \frac{s_{\beta 4}D + s_{\beta 5}D_4/R_a}{s_{\beta 54}},$$

These expressions prove the validity of formulas (3), (4) according to notations D, D_x, D_y and expressions $x_b = D_x - x_a$, $y_b = D_y - y_a$. The statement is proved.

From the formulas (1), (3), (4) it is obvious that to calculate the dynamic reactions at the support points A and B we must know the rotation angle, angular velocity and acceleration of OWM units and the driving forces moment in the hinges of these units. This is not a limitation of the obtained formulas but a consequence of the mechanics principles. The restrictions of the practical use of statement 2 are under consideration in the following notes.

Note 4. The case $s_{\beta 54} = 0$ is special because it doesn't let us use the formula (3) for the calculation x_a, y_a. It can occur at any point of time while WM is walking by different gaits, for example, $\mathbf{O_4A} \| \mathbf{O_5B}$. Calculations are performed by special algorithms at these times. This special case may occur while making a step discussed in sections 4 and 5 of this article.

Note 5. The case $s_{\beta 21} = 0$ is special because it doesn't help to use the formula (4) for the computation of x_b and y_b. To exclude this case from the consideration we assume that the first two units are not aligned along a single line in the three-point state, i.e. the two-unit leg is always in the configuration of the knee forward or backward in the reference state.

Statement 3. For OWM a projection on the axis O**k** (**k** is the normal to SP) of force reactions in the support points A and B are calculated by the formulas

$$z_a = (B_x g_s - B_y g_c)/(A_y B_x - A_x B_y), \quad z_b = (A_y g_c - A_x g_s)/(A_y B_x - A_x B_y), \qquad (8)$$

where $g_s = g\cos\beta \sum_{i=1}^{N} m_i d_i \sin\alpha_i + \sum_{i=1}^{N} \left(A_i \ddot{\alpha}_i - B_i \dot{\alpha}_i^2\right)$, $g_c = g\cos\beta \sum_{i=1}^{N} m_i d_i \cos\alpha_i + \sum_{i=1}^{N} \left(B_i \ddot{\alpha}_i + A_i \dot{\alpha}_i^2\right)$,

$A_i = \mathbf{i} \cdot I_{1i} \cdot \mathbf{k}$, $B_i = \mathbf{j} \cdot I_{1i} \cdot \mathbf{k}$, $I_{1i} = I_i - \mathbf{m_i OO_i} - \sum_{j,i} \mathbf{R_j m_j^c}$, m_i is the mass of the i-th augmented unit,

d_i is the center of mass of the i-th augmented unit, β is the angle of SP to the horizon, values I_i, $\mathbf{m_i}$, $\mathbf{m_i^c}$ are defined in article [6].

Proof. The moment of force $\mathbf{M_1}$ of the reference point O is calculated by the formula $\mathbf{M}_1 = \sum_{i=1}^{N} \left(I_{1i} \cdot \boldsymbol{\varepsilon_i} + \boldsymbol{\omega_i} \times I_{1i} \cdot \boldsymbol{\omega_i} + \boldsymbol{\omega_i \omega_i} \cdot \overline{I}_{1i}\right) - \mathbf{m_1^c} \times \mathbf{g} - \mathbf{R_{14}} \times \mathbf{F_{r4}} - \mathbf{R_{15}} \times \mathbf{F_{r5}}$ according to (6.26). This moment is acting on the shin of a two-unit leg from the side of SP. We will get $\mathbf{m_1^c} = \sum_{i=1}^{N} \mathbf{m_i} = \sum_{i=1}^{N} m_i d_i \mathbf{i_i}$ according to formulas (6.21), (6.2) for OWM. The contact of this OWM shin with the SP is a point, i.e. $\mathbf{i} \cdot \mathbf{M_1} = 0$, $\mathbf{j} \cdot \mathbf{M_1} = 0$. The rotation of all OWM units is parallel with SP , i.e. $\boldsymbol{\omega_i} = \dot{\alpha}_i \mathbf{k}$, $\boldsymbol{\varepsilon_i} = \ddot{\alpha}_i \mathbf{k}$. And by processing step there are $\mathbf{R_{14}} = \mathbf{OA}$, $\mathbf{R_{15}} = \mathbf{OB}$, $\mathbf{F_{r4}} = \mathbf{F_a}$, $\mathbf{F_{r5}} = \mathbf{F_b}$. Therefore,

$$0 = \mathbf{i} \cdot \mathbf{M_1} = \sum_{i=1}^{N} \left(\ddot{\alpha}_i \mathbf{i} \cdot I_{1i} \cdot \mathbf{k} + \dot{\alpha}_i^2 \mathbf{i} \cdot \mathbf{k} \times I_{1i} \cdot \mathbf{k} - m_i d_i \mathbf{i} \cdot \mathbf{i_i} \times \mathbf{g}\right) - \mathbf{i} \cdot \mathbf{OA} \times \mathbf{F_a} - \mathbf{i} \cdot \mathbf{OB} \times \mathbf{F_b},$$

$$0 = \mathbf{j} \cdot \mathbf{M_1} = \sum_{i=1}^{N} \left(\ddot{\alpha}_i \mathbf{j} \cdot I_{1i} \cdot \mathbf{k} + \dot{\alpha}_i^2 \mathbf{j} \cdot \mathbf{k} \times I_{1i} \cdot \mathbf{k} - m_i d_i \mathbf{j} \cdot \mathbf{i_i} \times \mathbf{g}\right) - \mathbf{j} \cdot \mathbf{OA} \times \mathbf{F_a} - \mathbf{j} \cdot \mathbf{OB} \times \mathbf{F_b}.$$

After elementary calculations

$$\overline{i} \cdot \mathbf{OA} \times \mathbf{F_a} = \mathbf{i} \times (A_x \mathbf{i} + A_y \mathbf{j}) \cdot \mathbf{F_a} = A_y \mathbf{k} \cdot \mathbf{F_a} = A_y z_a, \quad \mathbf{i} \cdot \mathbf{OB} \times \mathbf{F_b} = B_y z_b, \quad \mathbf{i} \cdot \mathbf{k} \times I_{1i} \cdot \mathbf{k} = -\mathbf{j} \cdot I_{1i} \cdot \mathbf{k},$$

$$\mathbf{j} \cdot \mathbf{OB} \times \mathbf{F_b} = \mathbf{j} \times (B_x \mathbf{i} + B_y \mathbf{j}) \cdot \mathbf{F_b} = -B_x \mathbf{k} \cdot \mathbf{F_b} = -B_x z_b, \quad \mathbf{j} \cdot \mathbf{OA} \times \mathbf{F_a} = -A_x z_a, \quad \mathbf{j} \cdot \mathbf{k} \times I_{1i} \cdot \mathbf{k} = \mathbf{i} \cdot I_{1i} \cdot \mathbf{k},$$

$$\mathbf{i} \cdot \mathbf{i_i} \times \mathbf{g} = \mathbf{i} \times \mathbf{i_i} \cdot \mathbf{g} = \mathbf{k} \cdot \mathbf{g} \sin\alpha_i, \quad \mathbf{j} \cdot \mathbf{i_i} \times \mathbf{g} = \mathbf{j} \times \mathbf{i_i} \cdot \mathbf{g} = \mathbf{k} \cdot \mathbf{g} \sin(\alpha_i - \pi/2) = -\mathbf{k} \cdot \mathbf{g} \cos\alpha_i, \quad \mathbf{k} \cdot \mathbf{g} = -g\cos\beta$$

we get the following system of equations relative to z_a, z_b.

$$\begin{cases} A_y z_a + B_y z_b = g_s, \quad g_s = \sum_{i=1}^{N}\left(A_i\ddot{\alpha}_i - B_i\dot{\alpha}_i^2 + gm_i d_i \sin\alpha_i \cos\beta\right), \\ A_x z_a + B_x z_b = g_c, \quad g_c = \sum_{i=1}^{N}\left(B_i\ddot{\alpha}_i + A_i\dot{\alpha}_i^2 + gm_i d_i \cos\alpha_i \cos\beta\right). \end{cases}$$

It is obvious that formula (8) is the solution of this system. The expression for I_{1i} is obtained from (6.24) taking into account $\mathbf{R_{1i}} = \mathbf{OO_i}$, $\mathbf{i}\cdot E\cdot\mathbf{k} = \mathbf{j}\cdot E\cdot\mathbf{k} = 0$. *The statement is proved.*

Corollary 1. In the three-point OWM or in the case of the equalities $I_{1i}^{xz} = I_{1i}^{yz} = 0$ for all i values z_a, z_b are calculated according to the formulas (8). In these formulas there are $g_s = mgr_x\cos\beta$, $g_c = mgr_y\cos\beta$, where m is the OWM mass, r_x, r_y are projections of the vector **OC** on the axis O**i**, O**j**; C is the OWM mass centre.

Proof. If OWM is stopped in the three-point state, then $\dot{\alpha}_i = \ddot{\alpha}_i = 0$ for all i. Therefore,

$$g_s = gm_c^x\cos\beta,\ g_c = gm_c^y\cos\beta, \text{ where } m_c^x = \sum_{i=1}^{N} m_i d_i \sin\alpha_i,\ m_c^y = \sum_{i=1}^{N} m_i d_i \cos\alpha_i .$$

If the elements I_{1i}^{xz}, I_{1i}^{yz} of the matrix I_{1i} are zero then $A_i = B_i = 0$ and $g_s = gm_c^x\cos\beta$, $g_c = gm_c^y\cos\beta$. According to (6.21) $\mathbf{m_1^c}$ is the static moment of the first subsystem relative to the point $O=O_1$. By definition [6] the system of bodies and its first subsystem are the same. In our case $\mathbf{m_1^c}$ is the static moment of OWM for the point O. Therefore $m_c^x = \mathbf{i}\cdot\mathbf{m_1^c}$ is the projection of $\mathbf{m_1^c}$ on the axis O**i**; $m_c^y = \mathbf{j}\cdot\mathbf{m_1^c}$ is the projection of $\mathbf{m_1^c}$ on the axis O**j**; $r_x = m_c^x/m$, $r_y = m_c^y/m$. *The corollary is proved.*

Note 6. The equations $A_y z_a + B_y z_b = m\,g\,r_x\cos\beta$, $A_x z_a + B_x z_b = m\,g\,r_y\cos\beta$ can be obtained from the conditions of static equilibrium of a rigid body. This rigid body is supported at three points on the plane located at angle β to the horizon. So in case of a three-point motionless state of OWM the corollar 1 can be proved using the equations of static equilibrium of a rigid body.

To calculate the dynamic response of OWM to the base point O we can use the following statement.

Statement 4. For OWM the projection of the reaction force at the reference point O on the axis O**k**, O**i**, O**j** are calculated by the formulas $z_o = mg\cos(\beta) - z_a - z_b$,

$$x_o = -\sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i s_i + \dot{\alpha}_i^2 c_i\right) - m\,\mathbf{i}\cdot\mathbf{g} - x_a - x_b,\quad y_o = \sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i c_i - \dot{\alpha}_i^2 s_i\right) - m\,\mathbf{j}\cdot\mathbf{g} - y_a - y_b,$$

where $s_i = \sin\alpha_i$, $c_i = \cos\alpha_i$.

Proof. According the formulas (7.1), (7.3) It is obvious that the reaction force of OWM at the support point O is calculated by the formula $\mathbf{F_o} = \sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i\mathbf{j_i} - \dot{\alpha}_i^2\mathbf{i_i}\right) - m_1\mathbf{g} - \mathbf{F_a} - \mathbf{F_b}$. Therefore, its projections x_o, y_o, z_o on the axis O**i**, O**j,** O**k** are calculated by the formulas

$$x_o = \mathbf{i}\cdot\mathbf{F_o} = \sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i\mathbf{i}\cdot\mathbf{j_i} - \dot{\alpha}_i^2\mathbf{i}\cdot\mathbf{i_i}\right) - m_1\mathbf{i}\cdot\mathbf{g} - x_a - x_b,$$

$$y_o = \mathbf{j}\cdot\mathbf{F_o} = \sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i\mathbf{j}\cdot\mathbf{j_i} - \dot{\alpha}_i^2\mathbf{j}\cdot\mathbf{i_i}\right) - m_1\mathbf{j}\cdot\mathbf{g} - y_a - y_b,$$

$$z_o = \mathbf{k}\cdot\mathbf{F_o} = \sum_{i=1}^{N} m_i d_i\left(\ddot{\alpha}_i\mathbf{k}\cdot\mathbf{j_i} - \dot{\alpha}_i^2\mathbf{k}\cdot\mathbf{i_i}\right) - m_1\mathbf{k}\cdot\mathbf{g} - z_a - z_b.$$

Hence, according to the equalities $\mathbf{k}\cdot\mathbf{i_i} = \mathbf{k}\cdot\mathbf{j_i} = 0$, $\mathbf{k}\cdot\mathbf{g} = -g\cos\beta$, $m_1 = m$, $\mathbf{i}\cdot\mathbf{i_i} = \mathbf{j}\cdot\mathbf{j_i} = \cos\alpha_i = c_i$, $\mathbf{i}\cdot\mathbf{j_i} = \cos(\alpha_i + \pi/2) = -\sin\alpha_i = -s_i$, $\mathbf{j}\cdot\mathbf{i_i} = \cos(\alpha_i - \pi/2) = \sin\alpha_i = s_i$ we will get the desired formula. *The statement is proved.*

Example 2. We will write formulas of the dynamic reactions at the support points A, B, O for OWM-5A in case of $s_{54} = \sin(\alpha_5 - \alpha_4) \neq 0$, $s_{21} = \sin(\alpha_2 - \alpha_1) \neq 0$, $I_{1i}^{xz} = I_{1i}^{yz}$ =0 for all i. According to the formula (3) we will receive

$$x_a = (D\cos\alpha_4 + D_4\cos\alpha_5/R)/s_{54}, \quad y_a = (D\sin\alpha_4 + D_4\sin\alpha_5/R)/s_{54},$$

$$x_b = (D_1\cos\alpha_2/a - D_2\cos\alpha_1/b)/s_{21} - x_a, \quad y_b = (D_1\sin\alpha_2/a - D_2\sin\alpha_1/b)/s_{21} - y_a,$$

where $D = \left[D_1\sin(\alpha_5 - \alpha_2)/a - D_2\sin(\alpha_5 - \alpha_1)/b\right]/s_{21} + D_5/R$. The left part of DE (1) has been obtained in example 1: $D_1 = J_1\ddot{\alpha}_1 + H_{21}\ddot{\alpha}_2 - h_{21}\dot{\alpha}_2^2 - G_1 - M_g$, $D_2 = H_{21}\ddot{\alpha}_1 + J_2\ddot{\alpha}_2 + h_{21}\dot{\alpha}_1^2 - G_2 + M_g - M_b$, $D_4 = J_4\ddot{\alpha}_4 - M_4$, $D_5 = J_5\ddot{\alpha}_5 - M_5$. The values z_a, z_b are calculated by the formulas (8) where according to the corollary 1 $g_s = g\cos\beta\sum_{i=1}^{2} m_i d_i \sin\alpha_i$, $g_c = g\cos\beta\sum_{i=1}^{2} m_i d_i \cos\alpha_i$. According to statement 4

$$x_o = -\sum_{i=1}^{2} m_i d_i\left(\ddot{\alpha}_i s_i + \dot{\alpha}_i^2 c_i\right) - m\,\mathbf{i}\cdot\mathbf{g} - x_a - x_b, \quad y_o = \sum_{i=1}^{2} m_i d_i\left(\ddot{\alpha}_i c_i - \dot{\alpha}_i^2 s_i\right) - m\,\mathbf{j}\cdot\mathbf{g} - y_a - y_b, \quad z_o = m\,g\cos\beta - z_a - z_b.$$

The values **i·g** , **j·g** are defined by the axes orientation O**i**, O**j** in inclined SP. If SP is horizontal ($\beta = 0$) then **i·g = j·g** = 0.

4. OWM DE with excluded relation reactions

The third equation of system (1) has not been used in the proof of statement 2. If we substitute the formulas (3) and (4) we get next statement.

Statement 5. The DE of OWM third unit with the excluded relation reactions has the form

$$b_1 D_1 + b_2 D_2 - D_3 + b_4 D_4 + b_s D_5/R_b = 0, \tag{9}$$

where

$$b_1 = \frac{b_s s_{\beta52} - R_5 s_{\gamma32}}{R_2 s_{\beta21}}, \quad b_2 = \frac{R_5 s_{\gamma31} - b_s s_{\beta51}}{R_3 s_{\beta21}}, \quad b_4 = \frac{R_4 s_{\beta53} - R_5 s_{\gamma53}}{R_a s_{\beta54}}, \quad b_s = \frac{R_4 s_{\beta43} - R_5 s_{\gamma43}}{s_{\beta54}}.$$

Proof. The third equation of system (1) has the form $D_3 = R_4(y_a c_{\beta3} - x_a s_{\beta3}) + R_5(y_b c_{\gamma3} - x_b s_{\gamma3})$. Using (4), we will exclude x_b, y_b from it. Then we get

$$D_3 = R_5 c_{\gamma3}(s_{\beta2}D_1/R_2 - s_{\beta1}D_2/R_3)/s_{\beta21} - R_5 s_{\gamma3}(c_{\beta2}D_1/R_2 - c_{\beta1}D_2/R_3)/s_{\beta21} +$$
$$+(R_5 s_{\gamma3} - R_4 s_{\beta3})(c_{\beta4}D + c_{\beta5}D_4/R_a)/s_{\beta54} + (R_4 c_{\beta3} - R_5 c_{\gamma3})(s_{\beta4}D + s_{\beta5}D_4/R_a)/s_{\beta54}.$$

Using (3), we will exclude x_a, y_a from it. Then we get

$$D_3 = -R_5[D_1(s_{\gamma3}c_{\beta2} - c_{\gamma3}s_{\beta2})/R_2 - D_2(s_{\gamma3}c_{\beta1} - c_{\gamma3}s_{\beta1})/R_3]/s_{\beta21} +$$
$$+(R_5 s_{\gamma3} - R_4 s_{\beta3})(c_{\beta4}D + c_{\beta5}D_4/R_a)/s_{\beta54} + (R_4 c_{\beta3} - R_5 c_{\gamma3})(s_{\beta4}D + s_{\beta5}D_4/R_a)/s_{\beta54}.$$

We will use the notation $s_{\gamma3i} = \sin(\alpha_3 + \gamma - \alpha_i - \beta_i) = s_{\gamma3}c_{\beta i} - c_{\gamma3}s_{\beta i}$ (i = 1,2) and do the elementary transformations. Then we get

$$D_3 = R_5(D_2 s_{\gamma31}/R_3 - D_1 s_{\gamma32}/R_2)/s_{\beta21} + \{[-R_5(s_{\beta4}c_{\gamma3} - c_{\beta4}s_{\gamma3}) + R_4(s_{\beta4}c_{\beta3} - c_{\beta4}s_{\beta3})]D +$$
$$+[-R_5(s_{\beta5}c_{\gamma3} - c_{\beta5}s_{\gamma3}) + R_4(s_{\beta5}c_{\beta3} - c_{\beta5}s_{\beta3})]D_4/R_a\}/s_{\beta54}.$$

Therefore,

$$D_3 = R_5(D_2 s_{\gamma31}/R_3 - D_1 s_{\gamma32}/R_2)/s_{\beta21} + [(R_4 s_{\beta53} - R_5 s_{\gamma53})D_4/R_a + (R_4 s_{\beta43} - R_5 s_{\gamma43})D]/s_{\beta54}.$$

We will select the D_i multipliers and substitute the expression (6) instead of D. Then we get

$$D_3 = \left(-\frac{R_5 s_{\gamma32}}{R_2 s_{\beta21}}\right)D_1 + \left(\frac{R_5 s_{\gamma31}}{R_3 s_{\beta21}}\right)D_2 + b_4 D_4 + b_s\left[\left(D_1 s_{\beta52}/R_2 - D_2 s_{\beta51}/R_3\right)/s_{\beta21} + D_5/R_b\right],$$

where $b_4 = (R_4 s_{\beta53} - R_5 s_{\gamma53})/(R_a s_{\beta54})$, $b_s = (R_4 s_{\beta43} - R_5 s_{\gamma43})/s_{\beta54}$. We will cast similar terms of D_1, D_2. Then we get the formula (9). *The statement is proved.*

Statement 6. OWM-5 DE in the three-point state with the excluded relation reactions can be represented as the following nonlinear ordinary differential equation of the second degree with respect to the generalized coordinate (GC) q

$$H(q)\ddot{q} + h(q)\dot{q}^2 + G(q) = (b_1 - b_2)M_g + (1 + b_2)M_b + (1 + b_4)M_4 + (1 + b_s/R_b)M_5 . \quad (10)$$

Where H, h, and G are given q functions and are calculated by the formulas

$$H = \sum_{k=1}^{5} b_k H_k \ , \ h = \sum_{k=1}^{5} b_k h_k \ , \ G = -\sum_{k=1}^{5} b_k G_k \ , \ G_k = g m_k d_k \mathbf{g} \cdot \mathbf{j_k} \sin\beta , \quad (11)$$

$$H_k = \sum_{i}^{k-1} H_{ki} f_{qi} + J_k f_{qk} + \sum_{i>k} H_{ik} f_{qi} \ , h_k = \sum_{i}^{k-1} \left(H_{ki} f_{qi}^q + h_{ki} f_{qi}^2 \right) + J_k f_{qk}^q + \sum_{i>k} \left(H_{ik} f_{qi}^q - h_{ik} f_{qi}^2 \right), \quad (12)$$

where $b_3 = -1$; $b_5 = b_s/R_b$; the absolute rotation angles of units are associated with GC q by the dependencies $\alpha_i = f_i(q)$ and $f_{qi} = df_i(q)/dq = d\alpha_i/dq$, $f_{qi}^q = d^2 f_i(q)/dq^2 = d^2\alpha_i/dq^2$. For example, the rotation angle of one contour unit may be used as q.

Proof. The left part of the k-th DE of the system (1) has the form [3]:

$$D_k = \sum_{i}^{k-1} \left(H_{ki}\ddot{\alpha}_i + h_{ki}\dot{\alpha}_i^2 \right) + J_k \ddot{\alpha}_k + \sum_{i>k} \left(H_{ik}\ddot{\alpha}_i - h_{ik}\dot{\alpha}_i^2 \right) - G_k - M_k + \sum_{i,k} M_i \ . \quad (13)$$

For contour units of relation equations (2) or as a result of kinematic analysis we will receive $\alpha_i = f_i(q)$ where q is GC of OWM-5 is the rotation angle of the dog [2], for example. After double differencing of functions $\alpha_i = f_i(q)$ of t we will get $\dot{\alpha}_i = f_{qi}\dot{q}$, $\ddot{\alpha}_i = f_{qi}\ddot{q} + f_{qi}^q \dot{q}^2$. We will substitute these expressions in (13). Then taking into account the notation (12), we will get

$$D_k = \sum_{i}^{k-1} \left[H_{ki} \left(f_{qi}\ddot{q} + f_{qi}^q \dot{q}^2 \right) + h_{ki} f_{qi}^2 \dot{q}^2 \right] + J_k \left(f_{qk}\ddot{q} + f_{qk}^q \dot{q}^2 \right) + \sum_{i>k} \left[H_{ik} \left(f_{qi}\ddot{q} + f_{qi}^q \dot{q}^2 \right) - h_{ik} f_{qi}^2 \dot{q}^2 \right] -$$

$$- G_k - M_k + \sum_{i,k} M_i = \left(\sum_{i}^{k-1} H_{ki} f_{qi} + J_k f_{qk} + \sum_{i>k} H_{ik} f_{qi} \right) \ddot{q} - G_k - M_k + \sum_{i,k} M_i +$$

$$+ \left[\sum_{i}^{k-1} \left(H_{ki} f_{qi}^q + h_{ki} f_{qi}^2 \right) + J_k f_{qk}^q + \sum_{i>k} \left(H_{ik} f_{qi}^q - h_{ik} f_{qi}^2 \right) \right] \dot{q}^2 = H_k \ddot{q} + h_k \dot{q}^2 - G_k - M_k + \sum_{i,k} M_i .$$

We will substitute the found expression for D_k into the formula (9) that can be written as $\sum_{k=1}^{5} b_k D_k = 0$, where $b_3 = -1$, $b_5 = b_s/R_b$. Then we will get

$$\sum_{k=1}^{5} b_k D_k = \sum_{k=1}^{5} b_k \left(H_k \ddot{q} + h_k \dot{q}^2 - G_k - M_k + \sum_{i,k} M_i \right) =$$

$$= \ddot{q} \sum_{k=1}^{5} b_k H_k + \dot{q}^2 \sum_{k=1}^{5} b_k h_k - \sum_{k=1}^{5} b_k G_k - \sum_{k=1}^{5} b_k \left(M_k - \sum_{i,k} M_i \right) = 0.$$

Hence, taking into account the notations (11), we will get DE (9) in the form $H\ddot{q} + h\dot{q}^2 + G = M$ where

$$M = b_1(M_1 - M_2) + b_2(M_2 - M_3) - (M_3 - M_4 - M_5) + b_4 M_4 + b_s M_5/R_b .$$

Taking into account that $M_1 = 0$, $M_2 = -M_g$, $M_3 = -M_b$, we will get

$$M = b_1 M_g + b_2(M_b - M_g) + M_b + M_4 + M_5 + b_4 M_4 + b_s M_5/R_b .$$

After casting similar terms of M_g, M_b, M_4, M_5 we will get the reqired form (10) of OWM-5 DE. *The statement is proved.*

Example 3. We write DE for OWM-5A in the case $s_{54} = \sin(\alpha_5 - \alpha_4) \neq 0$, $s_{21} = \sin(\alpha_2 - \alpha_1) \neq 0$. According to statement 6 for OWM-5A we will get the required DE as a form (10) where according to statement 5 we have

$b_1 = [L\sin(\alpha_3 - \alpha_2) + b_s\sin(\alpha_5 - \alpha_2)]/as_{21}$, $b_2 = [-L\sin(\alpha_3 - \alpha_1) - b_s\sin(\alpha_5 - \alpha_1)]/bs_{21}$,

$b_4 = 2L\sin(\alpha_5 - \alpha_3)/Rs_{54}$, $b_s = 2L\sin(\alpha_4 - \alpha_3)/s_{54}$.

Using the formulas (12) we will have

$H_1 = J_1 f_{q1} + H_{21} f_{q2}$, $H_2 = H_{21} f_{q1} + J_2 f_{q2}$, $h_1 = J_1 f_{q1}^q + H_{21} f_{q2}^q - h_{21} f_{q2}^2$, $h_2 = H_{21} f_{q1}^q + h_{21} f_{q1}^2 + J_2 f_{q2}^q$,

$H_k = J_k f_{qk}$, $h_k = J_k f_{qk}^q$, k = 3, 4, 5.

Using the formulas (11) we will have

$H = b_1 H_1 + b_2 H_2 - H_3 + b_4 H_4 + b_s H_5/R$, $h = b_1 h_1 + b_2 h_2 - h_3 + b_4 h_4 + b_s h_5/R$,

$G = -b_1 G_1 - b_2 G_2$, $G_1 = g m_1 d_1 \mathbf{g} \cdot \mathbf{j_1} \sin\beta$, $G_2 = g m_2 d_2 \mathbf{g} \cdot \mathbf{j_2} \sin\beta$.

5. OWM DE for the step forward

If the OWM has the equal lengths of single-unit legs (R_a=R_b), and while taking a step the absolute rotation angles of simple legs are equal (**O₄A||O₅B**), then the body (the 3rd unit) of OWM makes a translational displacement, i.e. $\alpha_3(t) = \text{const}$. Therefore, we will name the executed step as step forward (SF). It is easy to prove that $R_a = R_b$ is necessary for the implementation of SF.

According to note 4 for SF the DE (9) or (10) and formula (3) cannot be used. To study SF we can use statement 7.

Statement 7. If OWM has $R_a = R_b = R$ and by taking a step we have $\mathbf{O_4A} \| \mathbf{O_5B}$ then DE of its simple-unit leg is represented as

$$D_1 s_{\beta 42}/R_2 - D_2 s_{\beta 41}/R_3 + (D_4 + D_5) s_{\beta 21}/R = 0. \qquad (14)$$

Proof. If there are $\mathbf{O_4A} \| \mathbf{O_5B}$ then $s_{\beta 4} = s_{\beta 5}$, $c_{\beta 4} = c_{\beta 5}$ and after summing the equations of system (7) we get $x_a(s_{\beta 5} - s_{\beta 4}) + y_a(c_{\beta 4} - c_{\beta 5}) = D_4/R_a + D = 0$. We will substitute the expression (6) instead of D. Then we will get $(s_{\beta 52} D_1/R_2 - s_{\beta 51} D_2/R_3)/s_{\beta 21} + D_4/R_a + D_5/R_b = 0$. Taking into account $R = R_a = R_b$ and $s_{\beta 42} = s_{\beta 52}$, $s_{\beta 41} = s_{\beta 51}$ we will get the formula (14). *The statement is proved*.

If we consider OWM-5 then the equations (9) or (14) (for SF) are the only DE with excluded relation reactions. If N > 5 then the portable units must be added to DE (9) or (14), namely, we need to add equation 6 of the system (1) for OWM-6, we need to add the equation 6 and 7 of the system (1) for OWM-7, we need to add the equation 6, 7 and 8 of the system (1) for OWM-8. All of the added equations do not have dynamic reactions at the support points.

To study the dynamics of OWM-5 by taking a SF it can be used corollary 2.

Corollary 2. SF DE of OWM-5 can be represented as the following nonlinear differential equation of 2nd degree about the absolute angle q of the single-unit leg

$$H(q)\ddot{q} + h(q)\dot{q}^2 + G(q) = (b_1 - b_2)M_g + b_2 M_b + b_4(M_4 + M_5), \qquad (15)$$

where $b_1 = s_{q2}/R_2$, $b_2 = -s_{q1}/R_3$, $b_4 = s_{\beta 21}/R$, $s_{q1} = \sin(q - \alpha_1)$, $s_{q2} = \sin(q - \alpha_2)$.

Proof. The left part of DE (15) is proved on the basis of the DE (14) and is similar to the proof of statement 6. We will take the components $\sum_{i,k} M_i - M_k$ for D_k in (14) from (13) and transfer them to the right side of DE (14). Then DE (14) has the following form

$$H(q)\ddot{q} + h(q)\dot{q}^2 + G(q) = -s_{\beta 42}(M_2 - M_1)/R_2 + s_{\beta 41}(M_3 - M_2)/R_3 - s_{\beta 21}(-M_4 - M_5)/R.$$

If we use $s_{\beta 4i} = s_{qi}$ and take into account $M_1 = 0$, $M_2 = -M_g$, $M_3 = -M_b$ then we will get DE (15) from the last equation. *The corollary is proved*.

Corollary 3. SF ED of OWM-5A can be represented as

$$J\ddot{q} + \frac{1}{2}\frac{dJ}{dq}\dot{q}^2 + G = (f_{q1} - f_{q2})M_g + f_{q2}M_b + M_4 + M_5, \qquad (16)$$

where $J = J_4 + J_5 + J_1 f_{q1}^2 + J_2 f_{q2}^2 + 2d f_{q1} f_{q2} c_{21}$, $d = m_2 d_2 a$, $G = -f_{q1}G_1 - f_{q2}G_2$, $f_{q1} = Rs_{q2}/as_{21}$, $f_{q2} = -Rs_{q1}/bs_{21}$, $c_{21} = \cos(\alpha_2 - \alpha_1)$, $s_{21} = \sin(\alpha_2 - \alpha_1)$, a is the thigh length, b is the shin length of a two-unit leg, q is the absolute rotation angle of single-unit legs.

Proof. There are $\alpha_3 = \text{const}$, $\alpha_4 = \alpha_5 = q$ for SF. Hence, $f_{q3} = f_{q3}^q = 0$, $f_{q4} = f_{q5} = 1$, $f_{q4}^q = f_{q5}^q = 0$ and $H_3 = h_3 = 0$, $H_k = J_k$, $h_k = 0$ for k = 4, 5. Therefore, by using the data and results of examples 1, 3 according to the formulas (11), we will get $G = -b_1G_1 - b_2G_2$,

$$H = b_1H_1 + b_2H_2 + b_4J_4 + b_5J_5 = b_1\left(J_1f_{q1} + H_{21}f_{q2}\right) + b_2\left(H_{21}f_{q1} + J_2f_{q2}\right) + b_4J_4 + b_5J_5, \quad (17)$$

$$h = b_1h_1 + b_2h_2 = b_1(J_1f_{q1}^q + H_{21}f_{q2}^q - h_{21}f_{q2}^2) + b_2(H_{21}f_{q1}^q + h_{21}f_{q1}^2 + J_2f_{q2}^q), \quad (18)$$

where $b_1 = s_{q2}/a$, $b_2 = -s_{q1}/b$, $b_4 = b_5 = s_{21}/R$, $R_2 = a$ is the shin length, $R_3 = b$ is the thigh length of two-unit legs.

From the system (2) for OWM-5A we will get

$$\begin{cases} ac_1 + bc_2 = -Rc_q, & c_q = \cos(q), \quad s_q = \sin(q), \quad h = OD, \\ as_1 + bs_2 = h - Rs_q, & c_i = \cos(\alpha_i), \quad s_i = \sin(\alpha_i). \end{cases} \quad (19)$$

For deducing the calculation formulas f_{qi} (i = 1, 2) we will find q derivative of equations of system (19). Then we get the system of linear equations for f_{q1}, f_{q2}

$$\begin{cases} as_1f_{q1} + bs_2f_{q2} = -Rs_q \\ ac_1f_{q1} + bc_2f_{q2} = -Rc_q \end{cases}, \text{ and its determinant } \Delta = ab(s_1c_2 - c_1s_2) = -abs_{21}. \quad (20)$$

Therefore, the solution of this system

$$f_{q1} = \begin{vmatrix} -Rs_q & bs_2 \\ -Rc_q & bc_2 \end{vmatrix} / \Delta = \frac{Rb(s_2c_q - c_2s_q)}{-abs_{21}} = \frac{-R\sin(q-\alpha_2)}{-as_{21}} = \frac{Rs_{q2}}{as_{21}} = \frac{b_1}{b_4},$$

$$f_{q2} = \begin{vmatrix} as_1 & -Rs_q \\ ac_1 & -Rc_q \end{vmatrix} / \Delta = \frac{Ra(s_qc_1 - c_qs_1)}{-abs_{21}} = \frac{R\sin(q-\alpha_1)}{-bs_{21}} = \frac{-Rs_{q1}}{bs_{21}} = \frac{b_2}{b_4}.$$

We will divide DE (15) and b_4. Then the right side becomes the right side of equation (16), the formula for calculating G will take the required form, and the coefficient with $\ddot{q}$ will take the form $J = H/b_4 = f_{q1}\left(J_1f_{q1} + H_{21}f_{q2}\right) + f_{q2}\left(H_{21}f_{q1} + J_2f_{q2}\right) + J_4 + J_5$ according to (17). The last form will be equal to the required form due to $H_{21} = d\cdot\cos(\alpha_2 - \alpha_1)$. We will make a derivative J in q. Then taking into account $dH_{21}/dq = -d\cdot\sin(\alpha_2-\alpha_1)\cdot d(\alpha_2-\alpha_1)/dq = -h_{21}(f_{q2} - f_{q1})$ we will get

$$dJ/dq = f_{q1}^q\left(J_1f_{q1} + H_{21}f_{q2}\right) + f_{q1}\left(J_1f_{q1}^q - h_{21}(f_{q2}-f_{q1})f_{q2} + H_{21}f_{q2}^q\right) +$$
$$+ f_{q2}^q\left(H_{21}f_{q1} + J_2f_{q2}\right) + f_{q2}\left(-h_{21}(f_{q2}-f_{q1})f_{q2} + H_{21}f_{q1}^q + J_2f_{q2}^q\right) =$$
$$= f_{q1}(2J_1f_{q1}^q + 2H_{21}f_{q2}^q - 2h_{21}f_{q2}^2) + f_{q2}(2H_{21}f_{q1}^q + 2h_{21}f_{q1}^2 + 2J_2f_{q2}^q).$$

From (18) we get $h/b_4 = f_{q1}(J_1f_{q1}^q + H_{21}f_{q2}^q - h_{21}f_{q2}^2) + f_{q2}(H_{21}f_{q1}^q + h_{21}f_{q1}^2 + J_2f_{q2}^q)$, that proves the equality $\frac{h}{b_4} = \frac{1}{2}\frac{dJ}{dq}$. *The corollary is proved.*

Note 7. We can represent the left part of DE (10) in the form $H\ddot{q} + \frac{1}{2}\frac{dH}{dq}\dot{q}^2 + G$ as a consequence of statement 1 and 6. For single-moved mechanical systems the equality $H\ddot{q} + h\dot{q}^2 = H\ddot{q} + \frac{1}{2}\frac{dH}{dq}\dot{q}^2$, i.e. $h = \frac{1}{2}\frac{dH}{dq}$, is well-known [8]. The corollary 3 is proved to check the correctness of the obtained formulas.

According to note 4 it is impossible to calculate values x_a, y_a by the formulas (3) for SF ($\alpha_3 = \text{const}$). We can use statement 8.

Statement 8. The values x_a, y_a for SF ($\alpha_3 = \text{const}$) can be calculated by the formulas

$$x_a = (A_q c_{\beta 4} - c_\beta D_4/R)/(c_\beta c_{\beta 4} + c_\beta s_{\beta 4}), \quad y_a = (A_q s_{\beta 4} + s_\beta D_4/R)/(c_\beta c_{\beta 4} + c_\beta s_{\beta 4}), \tag{21}$$

where $A_q = D_3 + [D_1(s_\gamma c_{\beta 2} - c_\gamma s_{\beta 2})/R_2 + D_2(c_\gamma s_{\beta 1} - s_\gamma c_{\beta 1})/R_3]/s_{\beta 21}$,

$$s_\gamma = R_5 \sin(\gamma + \alpha_3), \quad c_\gamma = R_5 \cos(\gamma + \alpha_3), \quad s_\beta = s_\gamma - R_4 \sin(\beta_3 + \alpha_3), \quad c_\beta = R_4 \cos(\beta_3 + \alpha_3) - c_\gamma .$$

Proof. The third equation of system (1) was not used to derive SF DE. This equation has the form $R_4(y_a c_{\beta 3} - x_a s_{\beta 3}) + R_5(y_b c_{\gamma 3} - x_b s_{\gamma 3}) = D_3$. We substitute $x_b = D_x - x_a$, $y_b = D_y - y_a$ at this equation and we will get

$$(R_5 s_{\gamma 3} - R_4 s_{\beta 3})x_a + (R_4 c_{\beta 3} - R_5 c_{\gamma 3})y_a = D_3 + R_5(D_x s_{\gamma 3} - D_y c_{\gamma 3}).$$

We use the notation $s_\gamma = R_5 s_{\gamma 3}$, $c_\gamma = R_5 c_{\gamma 3}$, $s_\beta = s_\gamma - R_4 s_{\beta 3}$, $c_\beta = R_4 c_{\beta 3} - c_\gamma$. Then we will get the following system of two linear equations to calculate x_a, y_a together with the fourth equation of system (1)

$$\begin{cases} s_\beta x_a + c_\beta y_a = D_3 + s_\gamma D_x - c_\gamma D_y = A_q, \\ -s_{\beta 4} x_a + c_{\beta 4} y_a = D_4/R. \end{cases}$$

Formulas (21) are the solutions of this system. The determinant of this system is $\Delta = s_\beta c_{\beta 4} + c_\beta s_{\beta 4}$. For example, for OWM-5A $R_4 = R_5 = L$, 2L is the body length, $\alpha_3 = 0$, $\alpha_4 = q$, $\gamma = \pi$. Therefore, $s_\gamma = R_5 \sin\pi = 0$, $c_\gamma = R_5 \cos\pi = -L$, $s_\beta = 0$, $c_\beta = L - (-L) = 2L$ and $\Delta = 2L\sin q$, i.e. there are no problems with the calculation of dynamic reactions at the support points, if SF is limited by $0 < q < \pi$.

We substitute the known expressions that are introduced in the proof of statement 2 instead of D_x, D_y. Then we get

$$A_q = D_3 + s_\gamma(c_{\beta 2} D_1/R_2 - c_{\beta 1} D_2/R_3)/s_{\beta 21} - c_\gamma(s_{\beta 2} D_1/R_2 - s_{\beta 1} D_2/R_3)c_{\beta 5}]/s_{\beta 21} =$$

$$= D_3 + [D_1(s_\gamma c_{\beta 2} - c_\gamma s_{\beta 2})/R_2 + D_2(c_\gamma s_{\beta 1} - s_\gamma c_{\beta 1})/R_3]/s_{\beta 21}.$$

The statement is proved.

6. Algorithms for computing the OWM DE coefficients

To solve the dynamics of OWM on the basis of the received DE it is recommended to use known research methods of machines dynamics, for example, that were described in the fourth part of the book [8] on p. 223–235 or Chapter 2 of reference [9]. Moreover, we should write algorithms of calculations f_{qi}, f_{qi}^q, H(q) h(q) and G(q) and arrange all the formulas in order to use them in the numerical experiments. We must do it for each specific OWM and their steps. The dependences $\alpha_i = f_i(q)$ for the selected GC q in the mechanisms and machines theory are known as position functions (PF). and their q derivatives, i.e. the values f_{qi}, f_{qi}^q are called transfer functions of the 1st and 2nd degrees, respectively [8, 9]. We will consider examples of deducing the formulas for their computations.

Example 4. We do not have calculation formulas of values f_{q1}^q, f_{q2}^q so far. They are necessary for writing the algorithm for calculating H(q) and h(q) for OWM-5A. To do it we will find q derivatives of the equation system (20). Then we get the system of linear equations for f_{q1}^q, f_{q2}^q in the following form

$$\begin{cases} as_1 f_{q1}^q + bs_2 f_{q2}^q = -Rc_q - ac_1 f_{q1}^2 - bs_2 f_{q2}^2 = A, \\ ac_1 f_{q1}^q + bc_2 f_{q2}^q = Rs_q + as_1 f_{q1}^2 + bs_2 f_{q2}^2 = B. \end{cases}$$

We will write the solution of this system

$$f_{q1}^q = \begin{vmatrix} A & bs_2 \\ B & bc_2 \end{vmatrix} /\Delta = \frac{b(Ac_2 - Bs_2)}{-abs_{21}} = \frac{Bs_2 - Ac_2}{as_{21}}, \quad f_{q2}^q = \begin{vmatrix} as_1 & A \\ ac_1 & B \end{vmatrix} /\Delta = \frac{a(Bs_1 - Ac_1)}{-abs_{21}} = \frac{Ac_1 - Bs_1}{bs_{21}},$$

where

$$Bs_2 - Ac_2 = R(s_q s_2 + c_q c_2) + af_{q1}^2(s_1 s_2 + c_1 c_2) + bf_{q2}^2(s_2^2 + c_2^2) = Rc_{q2} + ac_{21} f_{q1}^2 + bf_{q2}^2 -$$

$$-Ac_1 + Bs_1 = R(s_q s_1 + c_q c_1) + af_{q1}^2(s_1^2 + c_1^2) + bf_{q2}^2(s_1 s_2 + c_1 c_2) = Rc_{q1} + af_{q1}^2 + bf_{q2}^2 c_{21}.$$

Taking into account the formulas derived in the proof of corollary 3 we will write down the algorithm for calculating H(q) h(q) for SF of OWM-5A on the level (G = 0).

$d = m_2 d_2 a = const$, $s_q = \sin(q)$, $c_q = \cos(q)$, $s_i = \sin(\alpha_i)$, $c_i = \cos(\alpha_i)$,

$s_{qi} = s_q c_i - c_q s_i$, $c_{qi} = s_q s_i + c_q c_i$, (i = 1, 2), $s_{21} = s_2 c_1 - c_2 s_1$, $c_{21} = s_1 s_2 + c_1 c_2$,

$H_{21} = dc_{21}$, $h_{21} = ds_{21}$,

$b_1 = s_{q2}/a$, $b_2 = -s_{q1}/b$, $b_4 = s_{21}/R$, $f_{q1} = b_1/b_4$, $f_{q2} = b_2/b_4$,

$f_{q1}^q = (Rc_{q2} + ac_{21}f_{q1}^2 + bf_{q2}^2)/(as_{21})$, $f_{q2}^q = -(Rc_{q1} + af_{q1}^2 + bf_{q2}^2 c_{21})/(bs_{21})$,

$H = b_1\left(J_1 f_{q1} + H_{21} f_{q2}\right) + b_2\left(H_{21} f_{q1} + J_2 f_{q2}\right) + b_4(J_4 + J_5)$,

$h = b_1(J_1 f_{q1}^q + H_{21} f_{q2}^q - h_{21} f_{q2}^2) + b_2(H_{21} f_{q1}^q + h_{21} f_{q1}^2 + J_2 f_{q2}^q)$.

Here it is considered to be that the PF $\alpha_1(q)$ and $\alpha_2(q)$ are known. It should be noted that the analytical forms of PF are very bulky and contain inverse trigonometric functions. The PF can be found as the solution of a system of nonlinear equations (19) or directly from the drawings of the relevant kinematic schemes by elementary geometric reasoning. The methods of deriving formulas for L PF are well known from the mechanisms and machines theory, for example in [8]. As an example, we will derive the PF $\alpha_1(q)$ and $\alpha_2(q)$ for the case of equal lengths of the thigh and shin of a two-unit leg. To do this, we will find the solution of system (19) when a = b.

$$\begin{cases} \cos\alpha_1 + \cos\alpha_2 = -R_a\cos q, \quad R_a = R/a, \quad h_a = h/a, \\ \sin\alpha_1 + \sin\alpha_2 = h_a - R_a\sin q. \end{cases} \qquad (22)$$

After squaring and summing equations of the system (22) we will get the equation $2 + 2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) = R_a^2 + h_a^2 - 2h_a R_a\sin q$. Hence, taking into account the symbols $a_1 = (R_a^2 + h_a^2)/2 - 1$, $a_2 = R_a h_a$ we will get $\cos(\alpha_2 - \alpha_1) = a_q = a_1 - a_2\sin q$. Therefore, if $\left|a_q\right| \le 1$, then $\alpha_2 - \alpha_1 = \arccos(a_q)$.

We will subtract the square of the second equation of system (22) from the square of the first one we will get

$$e = \cos^2\alpha_1 - \sin^2\alpha_1 + 2(\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2) + \cos^2\alpha_2 - \sin^2\alpha_2 =$$
$$= \cos 2\alpha_1 + \cos 2\alpha_2 + 2\cos(\alpha_1 + \alpha_2) = R_a^2(\cos^2 q - \sin^2 q) - h_a^2 + 2h_a R_a\sin q =$$
$$= R_a^2 - h_a^2 + 2h_a R_a\sin q - 2R_a^2\sin^2 q.$$

Taking into account $\cos 2\alpha_1 + \cos 2\alpha_2 = 2\cos[(2\alpha_1 + 2\alpha_2)/2]\cdot\cos[(2\alpha_1 - 2\alpha_2)/2]$ we will get

$$e = 2\cos(\alpha_1 + \alpha_2)\cdot\cos(\alpha_1 - \alpha_2) + 2\cos(\alpha_1 + \alpha_2) = \cos(\alpha_1 + \alpha_2)[2 + 2\cos(\alpha_1 - \alpha_2)].$$

And taking into account $\cos(\alpha_1 - \alpha_2) = a_1 - a_2\sin q = a_q$ we will get

$$e = 2\cos(\alpha_1 + \alpha_2)(1 + a_q) = 2a_o + 2a_2\sin q - 2R_a^2\sin^2 q,$$

where $a_o = (R_a^2 - h_a^2)/2$ and $\cos(\alpha_1 + \alpha_2) = b_q = (a_o + a_2\sin q - R_a^2\sin^2 q)/(1 + a_q)$. Therefore, if $\left|b_q\right| \le 1$, then $\alpha_1 + \alpha_2 = \arccos(b_q)$.

Thus, if $\left|a_q\right| \le 1$ and $\left|b_q\right| \le 1$, then the required SF are represented in the form

$$\alpha_1 = [\arccos(b_q) - \arccos(a_q)]/2, \quad \alpha_2 = [\arccos(b_q) + \arccos(a_q)]/2,$$

where, $a_q = a_1 - a_2\sin q$, $b_q = (a_o + a_2\sin q - R_a^2\sin^2 q)/(1 + a_q)$.

Example 5. If OWM makes a step that is different from SF then the expression for f_{qi}, f_{qi}^q are derived from the system (2). If we accept α_5 as GC, i.e., $q = \alpha_5$, then for calculating f_{q3}, f_{q4}, f_{q3}^q, f_{q4}^q from (2) we will get

$$\begin{cases} A_y - R_4 s_{\beta 3} - R_a s_{\beta 4} + R_5 s_{\gamma 3} + R_b s_{\beta 5} = B_y, \\ A_x - R_4 c_{\beta 3} - R_a c_{\beta 4} + R_5 c_{\gamma 3} + R_b c_{\beta 5} = B_x, \end{cases}$$

i.e.

$$\begin{cases} -R_4 s_{\beta 3} + R_5 s_{\gamma 3} - R_a s_{\beta 4} = B_y - A_y - R_b s_{\beta 5}, \\ -R_4 c_{\beta 3} + R_5 c_{\gamma 3} - R_a c_{\beta 4} = B_x - A_x - R_b c_{\beta 5}. \end{cases}$$

When we make double q differencing of the last system we will get

$$\begin{cases} (R_4 c_{\beta 3} - R_5 c_{\gamma 3}) f_{q3} + R_a c_{\beta 4} f_{q4} = R_b c_{\beta q}, \\ (R_4 s_{\beta 3} - R_5 s_{\gamma 3}) f_{q3} + R_a s_{\beta 4} f_{q4} = R_b s_{\beta q}, \end{cases}$$

$$\begin{cases} (R_4 c_{\beta 3} - R_5 c_{\gamma 3}) f_{q3}^q + R_a c_{\beta 4} f_{q4}^q = A, \\ (R_4 s_{\beta 3} - R_5 s_{\gamma 3}) f_{q3}^q + R_a s_{\beta 4} f_{q4}^q = B, \end{cases}$$

where $A = (R_4 s_{\beta 3} - R_5 s_{\gamma 3}) f_{q3}^2 + R_a s_{\beta 4} f_{q4}^2 - R_b s_{\beta q}$, $B = -(R_4 c_{\beta 3} - R_5 c_{\gamma 3}) f_{q3}^2 - R_a c_{\beta 4} f_{q4}^2 + R_b c_{\beta q}$, $s_{\beta q} = \sin(\beta_4 + q)$, $c_{\beta q} = \cos(\beta_4 + q)$. The solutions of these systems are elementary. To derive the formulas for values f_{q1}, f_{q2} from the system (2) we will get

$$\begin{cases} R_2 c_{\beta 1} + R_3 c_{\beta 2} = A_x - R_4 c_{\beta 3} - R_a c_{\beta 4}, \\ R_2 s_{\beta 1} + R_3 s_{\beta 2} = A_y - R_4 s_{\beta 3} - R_a s_{\beta 4}. \end{cases}$$

Here

$$\begin{cases} R_2 s_{\beta 1} f_{q1} + R_3 s_{\beta 2} f_{q2} = -R_4 s_{\beta 3} f_{q3} - R_a s_{\beta 4} f_{q4}, \\ R_2 c_{\beta 1} f_{q1} + R_3 c_{\beta 2} f_{q2} = -R_4 c_{\beta 3} f_{q3} - R_a c_{\beta 4} f_{q4}. \end{cases}$$

Here the latest system is to calculate values f_{q1}, f_{q2}, in which we use the previously derived formulas for f_{q3}, f_{q4}. When we make q differentiation of this system, we will get a system of two linear equations. The notation of the formulas and their simplification for a specific OWM is elementary but tedious analytical work.

7. Conclusion

Received OWM DE and formulas for calculating the dynamic reactions at the support points allow to organize the whole range of numerical experiments in the study of OWM including special cases and gaits.

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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ШАГАЮЩИХ АППАРАТОВ С ОДНОПОДВИЖНЫМ КОРПУСОМ

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Рассматриваются модели шагающих аппаратов (ША) с одноподвижным корпусом, кинематические схемы которых позволяют создавать ША с максимальной удельной грузоподъемностью и минимальным энергопотреблением приводов на реализацию заданного перемещения корпуса. Получены уравнения динамики (УД) таких ША в трехопорном состоянии. Эти УД в явном виде содержат кинематические, геометрические и инерционные параметры ОША-N, где N – любое натуральное число больше пяти. Количество математических операций в полученных УД минимально. УД представлены в двух видах: во-первых, в виде системы дифференциально-алгебраических уравнений, в которых дифференциальные уравнения содержат динамические реакции в опорных точках, а алгебраические – описывают геометрические связи опорных стоп с опорной поверхностью (ОП); во-вторых, в виде системы N дифференциальных уравнений второго порядка с исключенными реакциями связей. Формулы вычисления динамических реакций в опорных точках имеют максимально простой вид. Выведены формулы вычисления динамических реакций в опорных точках таких ША. Описаны алгоритмы решения задач динамики, возникающие при исследовании ходьбы рассматриваемых ША. Приведено четыре примера. В первом примере рассмотрен ША с одним силовым приводом.

Ключевые слова: шагающий аппарат, плоские модели, уравнения динамики, первая задача динамики, динамические реакции, движущие силы и моменты сил.

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