EXISTENCE OF LOWER AND UPPER SOLUTIONS IN REVERSE ORDER WITH RESPECT TO A VARIABLE IN A MODEL OF ACIDOGENESIS TO ANAEROBIC DIGESTION

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We prove existence of upper and lower solutions in reverse order with respect a part of the variables in a system of nonlinear ordinary differential equations modelling acidogenesis in anaerobic digestion. The corresponding existence theorems are established. The upper and lower solutions are constructed analytically, by defining semi-trivial solutions for each of the variables in the model. We introduce the concept of indicator semi-trivial solutions. Finally, we numerically solve the system supported by the Matlab software and matching the graphs of the numerical solutions with analytical solutions is found.

Keywords: upper-lower solutions; inverse order; system of nonlinear differential equations; anaerobic digestion.

Introduction

Anaerobic digestion (AD) is a microbial fermentation in the absence of oxygen results in a mixture of gases (mainly methane and carbon dioxide), known as "biogas" and an aqueous or "sludge" suspension containing the microorganisms responsible for degradation of organic matter. The raw material used primarily to be subjected to this treatment is any residual biomass that has a high moisture content, such as food scraps, leftover leaves and herbs to clean up a garden or orchard, livestock waste, sludge treatment plants urban wastewater and domestic sewage and industrial.

In practice, engineering becomes accustomed to consider three stages for solid waste or sludge (hydrolysis, acidogenic, methanogenic) and two for liquid waste (acidogenic and methanogenic) [1].

We consider the following system of nonlinear ordinary differential equations for the process AD [2]:

Balance of biomass

$$\frac{dX_1}{dt} = (\mu_1(S_1) - \alpha D) \; X_1 \quad \text{(acidogenic)} \tag{1}$$

$$\frac{dX_2}{dt} = (\mu_2(S_2) - \alpha D) \; X_2 \quad \text{(metanogenic)} \tag{2}$$

Balance of substrates

$$\frac{dS_1}{dt} = D \, (S_{1in} - S_1) - k_1 \; \mu_1(S_1) \; X_1 \quad \text{(acidogenic)} \tag{3}$$

$$\frac{dS_2}{dt} = D \, (S_{2in} - S_2) + k_2 \; \mu_1(S_1) \; X_1 - k_3 \; \mu_2(S_2) \; X_2 \quad \text{(metanogenic)} \tag{4}$$

Balance of alkalinity

$$\frac{dA}{dt} = D\left(A_{in} - A\right) \tag{5}$$

Carbon rate of change

$$\frac{dC}{dt} = D\left(C_{in} - C\right) + k_4\,\mu_1(S_1)\,X_1 + k_5\,\mu_2(S_2)\,X_2 - K_{L_a}[C + S_2 - A - K_H P_C]. \tag{6}$$

The expression $K_L a(C - K_H\,P_C)$ describes the molar flow rate of inorganic carbon from its liquid phase to its gas phase and the product $K_H\,P_C$ determines the concentration of dissolved oxygen in C.

Net rate of methane production

$$\frac{dF_M}{dt} = k_6\,\mu_2(S_2)\,X_2. \tag{7}$$

The kinetic comportment is nonlinear and occurs because of reaction rates, which are given by: Monod kinetics $\mu_1(S_1) = \mu_{1max}\frac{S_1}{S_1 + K_{S_1}}$ and Haldane kinetics $\mu_2(S_2) = \mu_{2max}\frac{S_2}{\frac{S_2^2}{K_{I2}} + S_2 + K_{S_2}}$. Bacterial rate represents yield related to both bioprocesses.

In this case the variables are:

$$S_1 := \text{Organic substrate concentration [g/l]}$$
$$X_1 := \text{Concentration of acidogenic bacteria [g/l]}$$
$$S_2 := \text{Volatile fatty acids concentration [mmol/l]}$$
$$X_2 := \text{Concentration of methanogenic bacteria [g/l]}$$
$$A := \text{Concentration of alkalinity [mmol/l]}$$
$$C := \text{Total inorganic carbon concentration [mmol/l]}$$
$$F_M := \text{Methane concentration [mmol/l d}^{-1}\text{]}.$$

The main objective is to build lower-upper solutions for the system formed by equations (1) and (3) in reverse order with respect to a part of variables with initial conditions on a given observation interval [3].

The corresponding equations:

$$\begin{aligned} u' &= f(x, u, v) \quad \text{in} \quad I, \\ v' &= g(x, u, v) \quad \text{in} \quad I, \end{aligned} \tag{8}$$

where $I = [a, b]$, $u = X_1$, $v = S_1$. Note that the definition of lower-upper solution (8) depends greatly on the properties of monotony of f and g. Therefore following notation of C.V. Pao in [4] and new results presented in [5], we can classify (8) according to their relative monotony, as follows:

1. Quasi-monotone systems: f and g are nondecreasing in v and u, respectively or f and g are nonincreasing in v and u, respectively.

2. Mixed quasi-monotone systems: either, f is nondecreasing in v and g is nonincreasing in u, or vice versa.

3. Nonquasi-monotone systems: the system does not fall in any of the previous cases.

Case 1 implies the existence of lower (u_*, v_*) and upper (u^*, v^*) solution with ordering in I

$$u_* \leq u^*, \quad v_* \leq v^*.$$

It is impossible to enforce quasi-monotonicity by a simple transformation for Case 2. Case 3 requires some regularity conditions imposed on f and g.

In the above system, the usual order $(X_{10} \leq X_1^0)$ is considered for the lower and upper solutions. For the variable S_1 the situation is different because of the opposite case $S_1^0 \leq S_{10}$. We consider the nonlinear equation [6]

$$u'(t) = f(t, u(t)), \quad t \in I = [0, T], \quad T > 0 \tag{9}$$

satisfying the condition [6]

$$g(u(0), u(T)) = 0, \tag{10}$$

where $f : I \times R \to R$ and $g : R^2 \to R$ are continuous functions. If $g(x, y) = x - c$ with $c \in R$, then (10) is the initial condition

$$u(0) = c.$$

Definition 1. [6]

- $\omega \in C^1(I)$ *is a lower solution to* (9) *if*

$$\omega'(t) \leq f(t, \omega(t)), \quad t \in I$$

and

$$\omega(t) \leq \beta(t), \quad t \in I \tag{11}$$

- $\beta \in C^1(I)$ *is upper solution to* (9) *if*

$$\beta'(t) \geq f(t, \beta(t)), \quad t \in I$$

and

$$\beta(t) \leq \omega(t), \quad t \in I \tag{12}$$

For $u, v \in C(I)$, $u \leq v$ define the set

$$[u, v] = \{\forall w \in C(I) \ : \ u(t) \leq w(t) \leq v(t), \text{ with } t \in I\}.$$

Definition 2. [6] *We say that* $\omega, \beta \in C^1(I)$ *are lower and upper coupled solutions to problem* (9), (10) *in direct order if* ω *is a lower solution and* β *is an upper solution to equation* (9), *with condition* (11) *and*

$$\max\{g(\omega(0), \omega(T)), g(\beta(0), \beta(T))\} \leq 0 \leq \min\{g(\beta(0), \beta(T)), g(\beta(0), \omega(T))\}.$$

Definition 3. [6] *We say that $\omega, \beta \in C^1(I)$ are lower and upper coupled solutions to problem (9), (10) in inverse order if ω is a lower solution and β is an upper solution to equation (9), with condition (12) and*

$$\max\{g(\omega(0), \omega(T)), g(\beta(0), \omega(T))\} \leq 0 \leq \min\{g(\beta(0), \beta(T)), g(\omega(0), \beta(T))\}.$$

Theorem 1. [6] *It is assumed that ω, β are lower and upper solutions coupled in inverse order to problem (9), (10). Additionally it is assumed that the functions*

$$h_\omega(x) := g(x, \omega(T)), \quad h_\beta := g(x, \beta(T))$$

are monotonic (both non-increasing or non-decreasing) in $[\beta(0), \omega(0)]$. Then there exists at least one solution of problem (1) − (3) in $[\beta, \omega]$.

The inverse order of lower and upper solutions for system (1) − (7) were not previously considered. The outline of this paper comprises the following stages: definition of trivial solutions for the complete system, construction of lower and upper solutions for a subsystem, study and formulation of the corresponding theorem for the existence of lower and upper solutions.

Definition 4. *A trivial solution of system (1) − (7) has the form*

$$
\begin{aligned}
&E_1(S_{1in}, 0), \ E_2(S_{1in}, 0, S_{2in}, 0), \ E_3(0, 0, S_{2in}) \\
&E_4(S_{1in}, 0, S_{2in}, 0, A_{in}), \ E_5(A_{in}), \ E_6(0, S_{2in}, 0)
\end{aligned}
\tag{13}
$$

where

$$X_1 = 0, \ X_2 = 0, \ S_1 = S_{1in}$$

$$S_2 = S_{2in}, \ A = A_{in}, \ C = C_{in}, \ F_M = 0$$

We do not consider other possibilities of trivial solutions in this document. A similar approach can be found in [7].

1. Main Results

1.1. Lower and Upper Solutions to the Initial Value Problem

We are interested in solutions of nonlinear system with initial conditions, which models the dynamics of biomass and substrate in the acidogenic.

$$\frac{dX_1}{dt} = \left(\mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D\right) X_1 \qquad \triangleq F(t, X_1(t), S_1(t)), \tag{14a}$$

$$\frac{dS_1}{dt} = D(S_{1in} - S_1) - k_1 \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} X_1 \qquad \triangleq G(t, X_1(t), S_1(t)), \tag{14b}$$

$$X_1(0) = c_1, \tag{14c}$$

$$S_1(0) = c_2. \tag{14d}$$

Where $t \in [0, T] = I$, with $T > 0$ and F, G are functions of class $C^0(I) = C(I)$, i.e., continuous functions in I.

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A. Define

$$f(S_1) = \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \quad \text{and} \quad g(S_1) = \mu_{1max} \frac{S_1}{S_1 + K_{S_1}}.$$

The function $g(S_1) = \mu_{1max} \frac{S_1}{S_1 + K_{S_1}}$ with $\mu_{1max}, K_{S_1} > 0$ is a monotonically increasing function such that $g(S_1) \longrightarrow \mu_{1max}$ as $S_1 \longrightarrow \infty$.

If $\mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \geq 0$ then it can be proved that $f(S_1) = O(g(S_1))$, i.e.,

$$\forall(S_1) \;\; \exists(c > 0) \quad \text{such that} \quad \|f(S_1)\| \leq c\|g(S_1)\|.$$

$$0 \leq \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \leq \mu_{1max} \frac{S_1}{S_1 + K_{S_1}}$$
$$\implies \left\| \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \right\| \leq 1 \cdot \left\| \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} \right\|$$
$$\implies \|f(S_1)\| \leq c \, \|g(S_1)\|$$
$$\implies \left\| \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \right\| \leq K.$$

If $\mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D < 0$ then $X_1 \longrightarrow 0$ and

$$\left\| \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \right\| \|X_1\| \leq K.$$

Condition **A** shows that the norm

$$\left\| \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} - \alpha D \right\|$$

is bounded.

We define the lower-upper solution in inverse order with respect to the variable S_1 and in direct order for X_1 as follows:

Definition 5. [Lower-upper solution] *A pair* $[(X_{10}, S_{10}), (X_1^0, S_1^0)]$ *is called*

(a) *a lower-upper solution of problem* (14)*, if the following conditions are satisfied*

$$(X_{10}, S_{10}) \in C^1(I), \qquad (X_1^0, S_1^0) \in C^1(I), \qquad t \in I$$

$$\dot{X}_{10} - F(t, X_{10}, S_1) \leq 0 \qquad \qquad \text{(lower)} \qquad \qquad \text{(15a)}$$

$$\dot{X}_1^0 - F(t, X_1^0, S_1) \geq 0 \;\; \text{in} \;\; I, \quad \forall S_1 \in [S_1^0, S_{10}], \quad \text{(upper)} \qquad \text{(15b)}$$

$$\dot{S}_{10} - G(t, X_1, S_{10}) \leq 0 \qquad \qquad \text{(lower in inverse order)} \quad \text{(15c)}$$

$$\dot{S}_1^0 - G(t, X_1, S_1^0) \geq 0 \;\; \text{in} \;\; I, \quad \forall X_1 \in [X_{10}, X_1^0] \quad \text{(upper in inverse order)} \quad \text{(15d)}$$
$$\text{with}$$

$$X_{10}(0) \leq c_2 \leq X_1^0(0), \;\; S_1^0(0) \leq c_1 \leq S_{10}(0) \qquad \text{(initial conditions);} \qquad \text{(15e)}$$

(b) a lower-lower solution of problem (14), if the following conditions are satisfied

$$\dot{X}_{10} - F(t, X_{10}, S_{10}) \leq 0 \quad in \quad I,$$
$$\dot{S}_0 - G(t, X_{10}, S_{10}) \leq 0 \quad in \quad I$$

with $S_1^0(0) \leq c_1 \leq S_{10}(0)$;

(c) an upper-upper solution of problem (14), if the following conditions are satisfied

$$\dot{X}_1^0 - F(t, X_1^0, S_1^0) \geq 0 \quad in \quad I,$$
$$\dot{S}_1^0 - G(t, X_1^0, S_1^0) \geq 0 \quad in \quad I$$

with $X_{10} \leq X_1^0, \quad S_1^0 \leq S_{10} \quad in \quad I$.

Definition 6. *The functions $\Phi(t, t_{S_{1i}}, \mu_{1max}, K_{S_1}, D, \alpha)$, $\Phi_1(t, t_{X_{1i}}, D, S_{1in}, k_1, \mu_{1max}, K_{S_1})$ are called semitrivial solutions of problem (14)*

if $\Phi(t, t_{S_{1i}}, \mu_{1max}, K_{S_1}, D, \alpha)$ is a solution of the ODE:

$$\dot{X}_1 = \left(\mu_{1max} \frac{t_{S_{1i}}}{t_{S_{1i}} + K_{S_1}} - \alpha D \right) X_1 \tag{16}$$

and $\Phi_1(t, t_{X_{1i}}, D, S_{1in}, k_1, \mu_{1max}, K_{S_1}, \alpha)$ is a solution of the following ODE:

$$\dot{S}_1 = D(S_{1in} - S_1) + k_1 \mu_{1max} \frac{S_1}{S_1 + K_{S_1}} t_{X_{1i}}. \tag{17}$$

Here $t_{S_{1i}}, i = 1, 2, 3$ and $t_{X_{1i}}, i = 1, 2, 3$ are the indicators of semitrivial solutions $\Phi(t, t_{S_{1i}}, \mu_{1max}, K_{S_1}, D, \alpha)$ and $\Phi_1(t, t_{X_{1i}}, D, S_{in}, k_1, \mu_{1max}, K_{S_1})$ respectively, defined by the following way:
If $S_1 = S_{1in}$, then $t_{S_{11}} = S_{1in}$;
If $S_1 = S_1^0$, then $t_{S_{12}} = S_1^0$ is an upper solution of problem (17);
If $S_1 = S_{10}$, then $t_{S_{13}} = S_{10}$ is a lower solution of problem (17);
If $X_1 = 0$, then $t_{X_{11}} = 0$;
If $X_1 = X_1^0$, then $t_{X_{12}} = X_1^0$ is an upper solution of problem (16);
If $X_1 = X_{10}$, then $t_{X_{13}} = X_{10}$ is a lower solution of problem (16).

From Definition 6, we obtain different types of semitriviales solutions of system (14):

$$\dot{X}_1 = \left(\mu_{1max} \frac{S_{1in}}{S_{1in} + K_{S_1}} - \alpha D \right) X_1, \tag{18a}$$
$$\dot{S}_1 = D(S_{1in} - S_1) \tag{18b}$$

for (X_1, S_1) with ordering of lower and upper solution

$$X_{10}(t) \leq X_1^0(t), \quad S_1^0(t) \leq S_{10}(t).$$

Theorem 2. *Assume that condition **A** is fulfilled and there exists a pair (X_{10}, S_1^0), (X_1^0, S_{10}) of lower-upper solutions of (14). Then there exists at least a solution of*

semitrivial equations (16), (17) *and there exists a solution* (X_1, S_1) *of system* (14) *such that*

$$
\begin{aligned}
X_{10}(t) \leq X_1(t) \leq X_1^0(t), \qquad t \in I \\
S_1^0(t) \leq S_1(t) \leq S_{10}(t), \qquad t \in I.
\end{aligned}
\tag{19}
$$

Proof. It is divided into two steps. First, we consider a modified problem and show that any solution of this problem is also a solution of the original problem and that it is between X_{10} and X_1^0, S_1^0 and S_{10} (reverse order) by means of differential inequalities. Second, a direct application of the Banach fixed point theorem shows that equations (16), (17) for the semitrivial solutions have at least one solution. Last result guarantees the existence of solution (X_1, S_1) to system (14).

Step 1. Introduce the space

$$
K \equiv [X_{10}, X_1^0] \times [S_1^0, S_{10}] \subset E \equiv C(I) \times C(I).
$$

K is a bounded closed convex set in E. Now, given (X_1, S_1), define the functions:

$$
f_{(X_1, S_1)}(t, u) = \begin{cases}
F(t, X_1^0) + X_1^0 & \text{if} \quad X_1^0 < u, \\
F(t, u) + u & \text{if} \quad X_{10} \leq u \leq X_1^0, \\
F(t, X_{10})X_{10} & \text{if} \quad u < X_{10},
\end{cases}
\tag{20}
$$

$$
g_{(X_1, S_1)}(t, v) = \begin{cases}
G(t, S_1^0) - S_1^0 & \text{if} \quad S_1^0 > v, \\
G(t, v) - v & \text{if} \quad S_1^0 \leq v \leq S_{10}, \\
G(t, S_{10}) - S_{10} & \text{if} \quad v > S_{10}.
\end{cases}
\tag{21}
$$

Define the mapping $T : K \to E$ by $T(X_1, S_1) = (u, v)$, where u and v are the solutions, respectively, of the modified problems

$$
(I) \qquad u^{'}(t) + u(t) = f_{(X_1, S_1)}(t, u(t)),
$$

$$
u(0) = c_1,
$$

$$
(II) \qquad v^{'}(t) - v(t) = g_{(X_1, S_1)}(t, v(t)),
$$

$$
v(0) = c_2
$$

that and reduced to (14), when $(t, u(t)) \times (t, v(t)) \in E$. We claim that any solution u of Eq. (I) is such that $X_{10}(t) \leq u(t) \leq X_1^0(t)$ for all $t \in I$, so that it is also a solution of (14a).

We shall prove that $u(t) \leq X_1^0(t)$ for all $t \in I$. The proof of the other inequality is similar.

If u is a solution of (I), then, by (20), for some $t \in I$ such that $(u - X_1^0)(t) > 0$, one has

$$
u'(t) = F(t, X_1^0) + u(t) - X_1^0 > \dot{X}_1^0
$$

so that

$$
(u - X_1^0)'(t) > 0
$$

for some $t \in I$ such that $(u - X_1^0)(t) > 0$. Therefore $u(t) > X_1^0$ for all $t \in I$, but this can not happen because it contradicts the definition of upper solution $u(t) \leq X_1^0$. Hence, there exists $t_1 \in I$ such that $(u - X_1^0)(t_1) \leq 0$.

We shall prove that $v(t) \geq S_1^0(t)$ for all $t \in I$. The proof of the other inequality is similar.

If v is a solution of (II), then, by (21), for some $t \in I$ such that $(v - S_1^0)(t) < 0$, one has
$$v'(t) = G(t, S_1^0) + v(t) + S_1^0 < \dot{S}_1^0$$
so that
$$(v - S_1^0)'(t) < 0$$
for some $t \in I$ such that $(v - S_1^0)(t) < 0$. Therefore $v(t) < S_1^0$ for all $t \in I$, but this can not happen because it contradicts the definition $v(t) \geq S_1^0$. Hence, there exists $t_2 \in I$ such that $(v - S_1^0)(t_2) \geq 0$.

Step 2. Consider equations (16), (17) for semitrivial solutions. Apply the Banach fixed point theorem as following:

Define $T_1 : C(I) \longrightarrow C(I)$ by

$$T_1 X_{10}(t) := c_1 + \int_0^t \left[\mu_{1max} \frac{S_1^0(s)}{S_1^0(s) + K_{S_1}} - \alpha D \right] X_{10}(s)ds, \quad X_{10} \in C(I).$$

If $\|X_{10}\| \leq M_1$ then

$$|T_1 X_{10} - T_1 X_{10}^*| \leq \left\| \mu_{1max} \frac{S_1^0}{S_1^0 + K_{S_1}} - \alpha D \right\| \|X_{10} - X_{10}^*\|.$$

The norm is bounded due to condition **A**

$$\left\| \mu_{1max} \frac{S_1^0}{S_1^0 + K_{S_1}} - \alpha D \right\| \leq K_1$$

so

$$|T_1 X_{10} - T_1 X_{10}^*| \leq K_1 \|X_{10} - X_{10}^*\| \quad \text{for every} \quad K_1 > 0.$$

Now define $T_2 : C(I) \longrightarrow C(I)$ by

$$T_2 X_1^0(t) := c_1 + \int_0^t \left[\mu_{1max} \frac{S_1^0(s)}{S_1^0(s) + K_{S_1}} - \alpha D \right] X_1^0(s)ds, \quad X_1^0 \in C(I).$$

If $\|X_1^0\| \leq M_2$ then

$$|T_2 X_1^0 - T_2 X_1^{0*}| \leq \left\| \mu_{1max} \frac{S_1^0}{S_1^0 + K_{S_1}} - \alpha D \right\| \|X_1^0 - X_1^{0*}\|.$$

The norm is bounded due to condition **A**

$$|T_2 X_1^0 - T_2 X_1^{0*}| \leq K_1 \|X_1^0 - X_1^{0*}\| \quad \text{for every} \quad K_1 > 0.$$

Consequently T_1, T_2 are contractive mappings from $\{X_{10}, X_1^0 \in C(I), \ \|X_{10}\| \leq M_1, \|X_1^0\| \leq M_2\}$ for every $M_1 > 1$ and for every $M_2 > 1$. Then T_1 and T_2 have a unique fixed point in $C(I)$.

Similarly, define $T_3 : C(I) \longrightarrow C(I)$ by

$$T_3 S_1^0(t) := c_2 + \int_0^t \left[D\left(S_{1in} - S_1^0(s)\right) - k_1 \mu_{1max} \frac{S_1^0(s)}{S_1^0(s) + K_{S_1}} \ X_{10}(s) \right] \ S_1^0(s)ds, \ S_1^0 \in C(I).$$

If $\|S_1^0\| \leq N_1$ then

$$|T_3 S_1^0 - T_3 S_1^{0^*}| \leq \left\| DS_{1in} - \left[D + k_1 \mu_{1max} \frac{1}{S_1^0 + K_{S_1}} X_{10} \right] S_1^0 \right\| \|S_1^0 - S_1^{0^*}\|.$$

Estimate the norm

$$\left\| D\left(S_{1in} - S_1^0\right) - \frac{k_1 \ \mu_{1max} \ S_1^0}{S_1^0 + K_{S_1}} \ X_{10} \right\|. \tag{22}$$

As $\|X_{10}\| \leq M_1, \ \forall \ S_1^0, S_1^{0^*} \in C(I)$

$$D\left(S_{1in} - S_1^0\right) - \frac{k_1 \ \mu_{1max} \ S_1^0}{S_1^0 + K_{S_1}} \ X_{10} - \left(D\left(S_{1in} - S_1^{0^*}\right) - \frac{k_1 \ \mu_{1max} \ S_1^{0^*}}{S_1^{0^*} + K_{S_1}} \ X_{10} \right) =$$

$$= D(S_1^{0^*} - S_1^0) + k_1 \ \mu_{1max} \ \left(\frac{S_1^{0^*}}{S_1^{0^*} + K_{S_1}} - \frac{S_1^0}{S_1^0 + K_{S_1}} \right) X_{10}$$

then

$$\left\| D(S_1^{0^*} - S_1^0) + k_1 \ \mu_{1max} \ \left(\frac{S_1^{0^*} - S_1^0}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right) X_{10} \right\| \leq$$

$$\leq \|D(S_1^{0^*} - S_1^0)\| + |k_1 \ \mu_{1max}| \ \left\| \frac{S_1^{0^*} - S_1^0}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right\| \|X_{10}\| =$$

$$= \|D(S_1^{0^*} - S_1^0)\| + |k_1 \ \mu_{1max}| \ \left\| \frac{1}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right\| \|S_1^{0^*} - S_1^0\|\|X_{10}\|.$$

Now estimate the norm

$$\left\| \frac{S_1^{0^*} - S_1^0}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right\|$$

taking

$$h(S_1^0) = \frac{1}{(S_1^0 + K_{S_1})(S_1^{0^*} + K_{S_1})}$$

as $S_1^0, S_1^{0^*} > 0$ and the constant $K_{S_1} > 0$. Then $h(S_1^0) < 1$, hence $\|h(S_1^0)\| \leq K_2$ so one has

$$\left\| D(S_1^{0^*} - S_1^0) + k_1 \ \mu_{1max} \ \left(\frac{S_1^{0^*} - S_1^0}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right) X_{10} \right\| \leq$$

$$\leq \|D(S_1^{0^*} - S_1^0)\| + |k_1 \ \mu_{1max}| \ \left\| \frac{1}{(S_1^{0^*} + K_{S_1})(S_1^0 + K_{S_1})} \right\| \|S_1^{0^*} - S_1^0\|\|X_{10}\| \leq$$

$$\leq (|D| + |k_1\mu_{1max} K_2 M_1|) \|S_1^{0^*} - S_1^0\| \leq (K_3 = \max\{|D|, |k_1\mu_{1max} K_2 M_1|\})\|S_1^{0^*} - S_1^0\|.$$

Then $\|T_3 S_1^0 - T_3 S_1^{0*}\| \le K_3 \|S_1^0 - S_1^{0*}\|$, for every $K_3 > 0$.

Now define $T_4 : C(I) \longrightarrow C(I)$ by

$$T_4 S_{10}(t) := c_2 + \int_0^t \left[D \left(S_{1in} - S_{10}(s) \right) - k_1 \ \mu_{1max} \frac{S_{10}(s)}{S_{10}(s) + K_{S_1}} \ X_{10}(s) \right] S_{10}(s) ds, S_{10} \in C(I).$$

If $\|S_{10}\| \le N_2$ then

$$|T_4 S_{10} - T_4 S_{10}^*| \le \left\| D S_{1in} - \left[D + k_1 \mu_{1max} \frac{1}{S_{10} + K_{S_1}} X_{10} \right] S_{10} \right\| \|S_{10} - S_{10}^*\|.$$

Estimate the norm

$$\left\| D \left(S_{1in} - S_{10} \right) - \frac{k_1 \ \mu_{1max} \ S_{10}}{S_{10} + K_{S_1}} \ X_{10} \right\|. \tag{23}$$

As $\|X_{10}\| \le M_1, \ \forall \ S_{10}, S_{10}^* \in C(I)$

$$D \left(S_{1in} - S_{10} \right) - \frac{k_1 \ \mu_{1max} \ S_{10}}{S_{10} + K_{S_1}} \ X_{10} - \left(D \left(S_{1in} - S_{10}^* \right) - \frac{k_1 \ \mu_{1max} \ S_{10}^*}{S_{10}^* + K_{S_1}} \ X_{10} \right) =$$

$$= D(S_{10}^* - S_{10}) + k_1 \ \mu_{1max} \ \left(\frac{S_{10}^*}{S_{10}^* + K_{S_1}} - \frac{S_{10}}{S_{10} + K_{S_1}} \right) X_{10}$$

then

$$\left\| D(S_{10}^* - S_{10}) + k_1 \ \mu_{1max} \ \left(\frac{S_{10}^* - S_{10}}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right) X_{10} \right\| \le$$

$$\le \|D(S_{10}^* - S_{10})\| + |k_1 \ \mu_{1max}| \ \left\| \frac{S_{10}^* - S_{10}}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right\| \|X_{10}\| =$$

$$= \|D(S_{10}^* - S_{10})\| + |k_1 \ \mu_{1max}| \ \left\| \frac{1}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right\| \|S_{10}^* - S_{10}\| \|X_{10}\|.$$

Now estimate the norm

$$\left\| \frac{S_{10}^* - S_{10}}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right\|$$

taking

$$h(S_{10}) = \frac{1}{(S_{10} + K_{S_1})(S_{10}^* + K_{S_1})}.$$

As $S_{10}, S_{10}^* > 0$ and the constant $K_{S_1} > 0$ then $h(S_{10}) < 1$, then $\|h(S_{10})\| \le K_4$. So one has

$$\left\| D(S_{10}^* - S_{10}) + k_1 \ \mu_{1max} \ \left(\frac{S_{10}^* - S_{10}}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right) X_{10} \right\| \le$$

$$\le \|D(S_{10}^* - S_{10})\| + |k_1 \ \mu_{1max}| \ \left\| \frac{1}{(S_{10}^* + K_{S_1})(S_{10} + K_{S_1})} \right\| \|S_{10}^* - S_{10}\| \|X_{10}\| \le$$

$$\le (|D| + |k_1 \ \mu_{1max} K_4 M_1|) \|S_{10}^* - S_{10}\| \le (K_5 = \max\{|D|, |k_1 \mu_{1max} K_4 M_1|\}) \|S_{10}^* - S_{10}\|.$$

So $\|T_4 S_{10} - T_4 S_{10}^*\| \le K_5 \|S_{10} - S_{10}^*\|$ for every $K_5 > 0$.

Consequently T_3, T_4 are contractive mappings from $\{S_1^0, S_{10} \in C(I), \quad \|S_1^0\| \leq N_1, \|S_{10}\| \leq N_2\}$ for every $N_1 > 1$ and for every $N_2 > 1$. Then T_3 and T_4 have a unique fixed point in $C(I)$.

We have proved that every mapping of T_1, T_2, T_3, T_4 has a unique fixed point in $C(I)$, this guarantees the existence of solutions $S_{10}, S_1^0, X_{10}, X_1^0$. This is sufficient to guarantee the existence of solutions X_1 and S_1 such that $X_{10} \leq X_1 \leq X_1^0$ and $S_1^0 \leq S_1 \leq S_{10}$.

$\square$

1.2. Semitrivial Solutions S_1 and X_1

Consider in system (14) the trivial solution $E_1(0, S_{1in})$ defined by (13) and search an analytical solution of linear equation (15d)

$$\frac{dS_1}{dt} - D(S_{1in} - S_1) = 0,$$
$$S_1 = S_{1in} - [S_{1in} - S_1(0)] \exp(-Dt). \tag{24}$$

When $S_{1in} > S_1(0)$ a graph of solution at $t \to \infty$ asymptotically decreases to a value S_{1in}. And for $S_{1in} < S_1(0)$ a graph of solution at $t \to \infty$ asymptotically increases to a value S_{1in}. However, we wish to demonstrate the method of upper and lower solutions. We take $\epsilon > 0$, consider let the solution

$$S_1 = S_{1in} - [S_{1in} - S_1(0)] \exp(-Dt) + \epsilon.$$

Thus

$$\frac{dS_1}{dt} - D(S_{1in} - S_1) \geq 0.$$
$$D(S_{1in} - S_1) \exp(-Dt) - D(S_{1in} - S_1) \exp(-Dt) + D\epsilon \geq 0.$$

Then $D\epsilon \geq 0$ for $D > 0$ and solution $S_1 = S_1^0$ is a upper solution of (18b). If $\epsilon < 0$, then S_{10} is a lower solution of (18b).
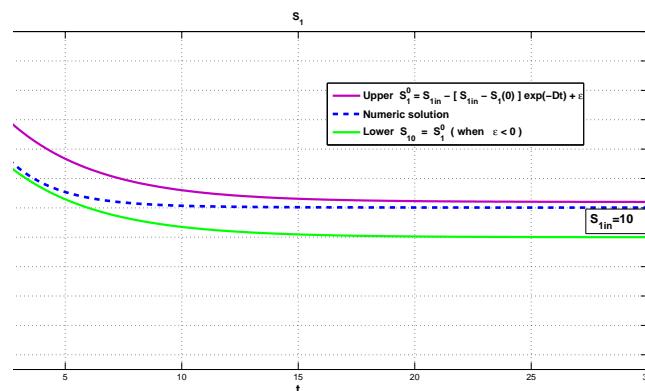
Fig. 1. Tending of a graph to S_{1in} as $t \to \infty$. The parameters for simulation from [2]

Fig. 1 shows that the solution tends to the value S_{1in} as $t \to \infty$. In the process, this represents that bacteria feed on the substrate S_1 over time.

The solution to S_1^0

Parameter	Value	Units	SD
$S_1(0)$	5	[g/l]	
D	0,395	[d^{-1}]	0,135
S_{1in}	10	[g/l]	6,4

Now we look for a semi-trivial upper-solution X_1^0 for equations (15b) and (24). From Definition 5

$$\dot{X_1}^0 - \left(\mu_{1max} \frac{S_1^0}{S_1^0 + K_{S_1}} - \alpha D \right) X_1^0 \geq 0.$$
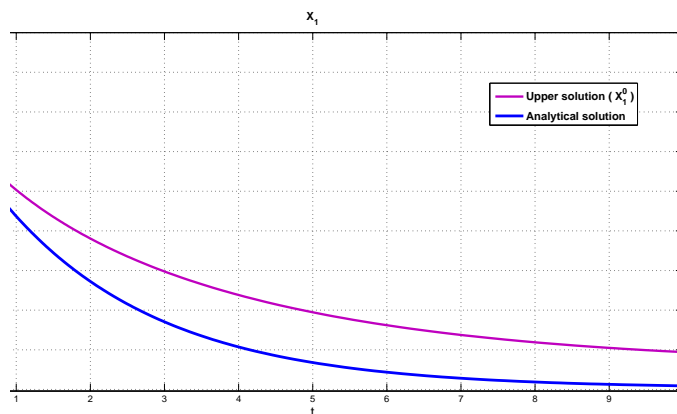
Taking (16) and (24) in (15b), with $c_8 = S_{1in} - S_1(0)$ we obtain

$$\implies \dot{X_1}^0 - \left(\mu_{1max} \frac{S_{1in} - c_8 \exp(-Dt)}{S_{1in} - c_8 \exp(-Dt) + K_{S_1}} - \alpha D \right) X_1^0 = 0$$

then

$$X_1^0 = c_9 * \frac{[c_8 \exp(-Dt)(S_{1in} + K_{S_1} - c_8 \exp(-Dt))]^{\frac{\mu_{1max} S_{1in}}{D(S_{1in} + K_{S_1})}}}{[S_{1in} + K_{S_1} - c_8 \exp(-Dt)]^{\frac{\mu_{1max}}{D}}} * \exp(-\alpha\ D\ t)$$

and a graphic representation is Fig. 2 shows that the upper solution X_1^0 presents *condition washout*.

Fig. 2. The graph of the solution X_1 tending to 0 as $t \to \infty$. The parameter values are taken from Table, with initial condition $X_1(0) = 0,5$

Conclusions

In this paper we presented a study of existence of lower and upper solutions of a system of ordinary differential equations modelling acidogenic stage process of AD. Inverse order of

lower and upper solutions with respect to variables was considered. We are well aware that this is only the first step in the complete study of the problem. The next step is to consider the V.M. Matrosov comparison principle to explore the global stability of solutions and a complete study of bifurcation (here readers may refer to monographs [8, 9]).

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СУЩЕСТВОВАНИЕ НИЖНИХ И ВЕРХНИХ РЕШЕНИЙ В ОБРАТНОМ ПОРЯДКЕ ПО ОТНОШЕНИЮ К ПЕРЕМЕННОЙ В МОДЕЛИ АЦИДОГЕНЕЗА ДЛЯ АНАЭРОБНОГО СБРАЖИВАНИЯ

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Доказано существование верхних и нижних решений в относительно части переменных в системе нелинейных обыкновенных дифференциальных уравнений, моделирующих ацидогенез в анаэробном сбраживании в задаче метанообразования. Верхние и нижние решения строятся аналитически, и установлены соответствующие теоремы существования.

Ключевые слова: верхние и нижния решения; обратный порядок; системы нелинейных дифференциальных уравнений; анаэробное сбраживание.

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