MSC 60H30

STRONGLY CONTINUOUS OPERATOR SEMIGROUPS. ALTERNATIVE APPROACH

A.A. Zamyshlyaeva, South Ural State University, Chelyabinsk, Russian Federation, alzama@mail.ru

Inheriting and continuing the tradition, dating back to the Hill–Iosida–Feller–Phillips–Miyadera theorem, the new way of construction of the approximations for strongly continuous operator semigroups with kernels is suggested in this paper in the framework of the Sobolev type equations theory, which experiences an epoch of blossoming. We introduce the concept of relatively radial operator, containing condition in the form of estimates for the derivatives of the relative resolvent, the existence of C_0-semigroup on some subspace of the original space is shown, the sufficient conditions of its coincidence with the whole space are given. The results are very useful in numerical study of different nonclassical mathematical models considered in the framework of the theory of the first order Sobolev type equations, and also to spread the ideas and methods to the higher order Sobolev type equations.

Keywords: Sobolev type equation, strongly continuous semigroups of operators with kernals, approximations of semigroups.

Introduction

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, operator $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$, function $f(\cdot) : \mathbb{R} \to \mathcal{F}$. Consider the Cauchy problem

$$u(0) = u_0 \tag{1}$$

for the operator-differential equation

$$L\,\dot{u} = Mu + f. \tag{2}$$

If the operator L is continuously invertible, then the equation (2) can be reduced to a pair of equivalent equations

$$\dot{u} = Su + h, \quad \dot{g} = Tg + f. \tag{3}$$

Here the operators $S = L^{-1}M \in \mathcal{C}l(\mathcal{U})$, $\text{dom}\ S = \text{dom}\ M$, $T = ML^{-1} \in \mathcal{C}l(\mathcal{F})$, $\text{dom}\ T = L[\text{dom}\ M]$, the function $h = L^{-1}f : \mathbb{R} \to \mathcal{U}$. It is convenient to consider the equation (3) in the frame of the equation

$$\dot{v} = Av + z \tag{4}$$

on the Banach space $\mathcal{V}$. Here $A : \text{dom}\ A \to \mathcal{V}$, $\overline{\text{dom}\ A} = \mathcal{V}$, $z(\cdot) : \mathbb{R} \to \mathcal{V}$.

The Cauchy problem

$$v(0) = v_0, \quad v_0 \in \text{dom}\ A \tag{5}$$

for the homogeneous equation

$$\dot{v} = Av \tag{6}$$

is completely studied with the help of the semigroups theory. The main result of the classical semigroups theory [1] is a theorem of Hill–Iosida–Feller–Phillips–Miyadera (the HIFPM theorem), establishing a bijection between the resolving semigroup of the homogeneous equation (6) and the operator A, called the infinitesimal generator of a semigroup. The criterion for the operator A being the infinitesimal generator of a semigroup (or generating the semigroup) are some conditions on the resolvent $R_\mu(A) = (\mu I - A)^{-1}$ of the operator A. Depending on these conditions, operator A generates the analytical group, analytical semigroup or strongly continuous (C_0) semigroup.

The theory of degenerate operator semigroups developed by G.A.Sviridyuk and his disciples generalizes these results to the case of the Sobolev type equations [2–6]. It also consists of three parts: analytical groups, analytical semigroups and, finally, strongly continuous semigroups with kernels. We suggest the alternative (in comparison to [7]) method of construction of C_0-semigroup for the equation (2). To our opinion, these results are very useful for the numerical modelling of different processes based on the first order Sobolev type equations and to spread methods to the higher order Sobolev type equations [8].

1. Relatively radial operators

Following [2, 7], introduce the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\}$ and the L-spectrum $\sigma^L(M) = \overline{\mathbb{C}} \setminus \rho^L(M)$ of operator M. The operator functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called L-resolvent, right and left L-resolvents of operator M.

Definition 1. *The operator M is called radial with respect to operator L (shortly, L-radial), if*
(i) $\exists a \in \mathbb{R} \quad \forall \mu > a \quad \mu \in \rho^L(M)$
(ii) $\exists K > 0 \quad \forall \mu > a \quad \forall n \in \mathbb{N}$

$$\max\{\|\frac{1}{n!}\frac{d^n}{d\mu^n}R^L_\mu(M)\|_{\mathcal{L}(\mathcal{U})}, \|\frac{1}{n!}\frac{d^n}{d\mu^n}L^L_\mu(M)\|_{\mathcal{L}(\mathcal{F})}\} \le \frac{K}{(\mu - a)^{n+1}}$$

Remark 1. Without loss of generality one can put $a = 0$ in definiton1.

Remark 2. If there exists the operator $L^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})$, then operator M is L-radial exactly, when the operator $L^{-1}M \in \mathcal{C}l(\mathcal{U})$ (or, equivalently, the operator $ML^{-1} \in \mathcal{C}l(\mathcal{F})$) is radial.

Set $\mathcal{U}^0 = \ker L \quad \mathcal{F}^0 = \ker L^L_\mu(M))$. By $L_0 \quad (M_0)$ denote restriction of the operator $L \quad (M)$ to lineal $\mathcal{U}^0 \quad (\mathrm{dom}\, M_0 = \mathcal{U}^0 \cap \mathrm{dom}\, M)$.

Definition 2. *Weakly L-radial operator is an operator M for which condition (i) is satisfied as well as condition (ii) when $n = 1$ in Definition 1.*

Theorem 1. [2] *Let the operator M be weakly L-radial. Then:*
(i) any vector $\varphi \in \ker L \setminus \{0\}$ does not have M-adjoint vectors;
(ii) $\ker R^L_\mu(M) \cap \mathrm{im}\, R^L_\mu(M) = \{0\}, \quad \ker L^L_\mu(M) \cap \mathrm{im}\, L^L_\mu(M) = \{0\}$.
(iii) there exists the operator $M_0^{-1} \in \mathcal{L}(\mathcal{F}^0;\mathcal{U}^0)$.

By $\mathcal{U}^1 \quad (\mathcal{F}^1)$ denote the closure of the lineal $\mathrm{im}\, R^L_\mu(M) \quad (\mathrm{im}\, L^L_\mu(M))$ by norm of the space $\mathcal{U} \quad (\mathcal{F})$.

Lemma 1. [2] *Let the operator M be weakly L-radial. Then*
(i) $\lim\limits_{\mu\to+\infty} \mu R^L_\mu(M)u = u \quad \forall u \in \mathcal{U}^1$;
(ii) $\lim\limits_{\mu\to+\infty} \mu L^L_\mu(M)f = f \quad \forall f \in \mathcal{F}^1$.

By $\tilde{\mathcal{U}} \quad (\tilde{\mathcal{F}})$ denote the closure of the lineal $\mathcal{U}^0 \dot{+}\, \mathrm{im}\, R^L_{(\mu,p)}(M) \quad (\mathcal{F}^0 \dot{+}\, \mathrm{im}\, L^L_{(\mu,p)}(M))$ by norm of the space $\mathcal{U} \quad (\mathcal{F})$. Obviously, $\mathcal{U}^1 \quad (\mathcal{F}^1)$ is the subspace in $\tilde{\mathcal{U}} \quad (\tilde{\mathcal{F}})$.

Lemma 2. *Let the operator M be weakly L-radial. Then* $\tilde{\mathcal{U}} = \mathcal{U}^0 \oplus \mathcal{U}^1, \quad \tilde{\mathcal{F}} = \mathcal{F}^0 \oplus \mathcal{F}^1$.

2. The resolving operator semigroups

Consider two equivalent forms of the equation (2)

$$R_\alpha^L(M)\dot{u} = (\alpha L - M)^{-1} M u, \tag{7}$$

$$L_\alpha^L(M)\dot{f} = M(\alpha L - M)^{-1} f \tag{8}$$

as concrete interpretations of the equation

$$A\dot{v} = Bv, \tag{9}$$

defined on a Banach space $\mathcal{V}$, where the operators $A, B \in \mathcal{L}(\mathcal{V})$

Definition 3. *The vector-function $v \in C(\overline{\mathbb{R}_+}; \mathcal{V})$, differentiable on $\mathbb{R}_+$ and satisfying (9) is called a solution of the equation (9).*

A little away from the standard [1], following [7] define

Definition 4. *The mapping $V^\cdot \in C(\overline{\mathbb{R}_+}; \mathcal{L}(\mathcal{V}))$ is called a semigroup of the resolving operators (a resolving semigroup) of the equation (9), if*
(i) $V^s V^t v = V^{s+t} v$ for all $s, t \geq 0$ and any v from the space $\mathcal{V}$;
(ii) $v(t) = V^t v$ is a solution of the equation (9) for any v from a dense in $\mathcal{V}$ set.
The semigroup is called uniformly bounded, if

$$\exists C > 0 \quad \forall t \geq 0 \quad \|V^t\|_{\mathcal{L}(\mathcal{V})} \leq C.$$

Theorem 2. *Let the operator M be L-radial. Then there exists a uniformly bounded and strogly continuous resolving semigroup of the equation (7) ((8)), treated on the subspace $\tilde{\mathcal{U}}$ $(\tilde{\mathcal{F}})$, presented in the form:*

$$U^t = s - \lim_{k \to +\infty} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} R_\mu^L(M)\right)\Bigg|_{\mu = \frac{k}{t}},$$

$$(F^t = s - \lim_{k \to +\infty} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} L_\mu^L(M)\right)\Bigg|_{\mu = \frac{k}{t}}).$$

Proof. Denote the following families of operators:

$$U_k^t = \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} R_\mu^L(M)\right)\Bigg|_{\mu = \frac{k}{t}}$$

Note that

$$\forall u \in \mathcal{U}^0 \qquad U_k^t u = 0. \tag{10}$$

Since the operator M is L-radial, approximations of are uniformly bounded by a constant K from Definition 1:

$$\|U_k^t\|_{\mathcal{L}(\mathcal{U})} \leq K \quad \forall t \in \mathbb{R}_+ \quad \forall k \in \mathbb{N}. \tag{11}$$

Let us take $u \in \operatorname{dom} M$ and find the derivative

$$\frac{d}{dt} U_k^t u = \frac{d}{dt} \left(\frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} R_\mu^L(M)\right)\Bigg|_{\mu = \frac{k}{t}} u \right) =$$

$$= \frac{(-1)^{k-1}}{(k-1)!} k^k \left(-k \frac{1}{t^{k+1}} \frac{d^{k-1}}{d\mu^{k-1}} R^L_\mu(M) - \frac{k}{t^{k+2}} \frac{d^k}{d\mu^k} R^L_\mu(M) \right) \Big|_{\mu=\frac{k}{t}} u =$$

$$= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^{k+1} \left(-\frac{d^{k-1}}{d\mu^{k-1}} R^L_\mu(M) + \frac{k}{t} \frac{d^{k-1}}{d\mu^{k-1}} \left(R^L_\mu(M)\right) R^L_\mu(M) \right) \Big|_{\mu=\frac{k}{t}} u =$$

$$= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^{k+1} \frac{d^{k-1}}{d\mu^{k-1}} \left(R^L_\mu(M)\right) \left(\frac{k}{t} R^L_\mu(M) - I \right) \Big|_{\mu=\frac{k}{t}} u =$$

$$= \frac{k}{t} U^t_k \left(\frac{k}{t} L - M \right)^{-1} Mu$$

Thus,

$$\frac{d}{dt} U^t_k u = U^t_k \Big(L - \frac{t}{k} M \Big)^{-1} Mu \qquad \forall u \in \mathrm{dom}\, M. \tag{12}$$

Now let $u \in \ \mathrm{im} R^L_\mu(M)$, i.e. $u = R^L_\beta(M)v$ in some $\beta > 0$ и $v \in \mathcal{U}$. Proof, then

$$\lim_{t\to 0+} U^t_k u = u. \tag{13}$$

Make the change $\mu = \frac{k}{t}$.

$$\lim_{t\to 0+} U^t_k u = \lim_{\mu\to+\infty} \frac{(-1)^{k-1}}{(k-1)!} \mu^k \frac{d^{k-1}}{d\mu^{k-1}} R^L_\mu(M) u =$$

$$= \lim_{\mu\to+\infty} \sum_{m=1}^{k-1} \left(\frac{(-1)^m}{m!} \mu^{m+1} \frac{d^m}{d\mu^m} R^L_\mu(M) - \frac{(-1)^{m-1}}{(m-1)!} \mu^m \frac{d^{m-1}}{d\mu^{m-1}} R^L_\mu(M) \right) u + \lim_{\mu\to+\infty} \mu R^L_\mu(M) u =$$

$$= \lim_{\mu\to+\infty} \sum_{m=1}^{k-1} \frac{(-1)^{m-1}}{(m-1)!} \mu^m \left(-\frac{\mu}{m} \frac{d^m}{d\mu^m} R^L_\mu(M) - \frac{d^{m-1}}{d\mu^{m-1}} R^L_\mu(M) \right) u + \lim_{\mu\to+\infty} \mu R^L_\mu(M) u =$$

$$= \lim_{\mu\to+\infty} \sum_{m=1}^{k-1} \frac{(-1)^{m-1}}{(m-1)!} \mu^m \frac{d^{m-1}}{d\mu^{m-1}} (R^L_\mu(M))(\mu R^L_\mu(M) - I) u + \lim_{\mu\to+\infty} \mu R^L_\mu(M) u. \tag{14}$$

Due to $\| \frac{-1}{(m-1)!} \mu^m \frac{d^{m-1}}{d\mu^{m-1}} (R^L_\mu(M)) \|_{\mathcal{L}(\mathcal{U})} \le K$ for any $\mu > 0$ under L-radiality of the operator M, we get

$$\left\| \sum_{m=1}^{k-1} \frac{(-1)^{m-1}}{(m-1)!} \mu^m \frac{d^{m-1}}{d\mu^{m-1}} (R^L_\mu(M))(\mu R^L_\mu(M) - I) u \right\|_{\mathcal{U}} \le$$

$$\le (k-1) K \| \mu R^L_\mu(M) u - u) \|_{\mathcal{U}} \to 0, \quad \mu \to +\infty.$$

The second limit in (14) is equal to u.

Now for the same vector u consider the difference

$$U^t_k u - U^t_l u = \int\limits_0^t \frac{d}{ds} \left(U^{t-s}_l U^s_k u \right) ds =$$

$$= \int\limits_0^t \left(U^{t-s}_l U^s_k \Big(L - \frac{s}{k} M \Big)^{-1} M - U^{t-s}_l U^s_k \Big(L - \frac{t-s}{l} M \Big)^{-1} M \right) u ds =$$

$$= \int\limits_0^t U_l^{t-s} U_k^s \left(\frac{s}{k} - \frac{t-s}{l}\right) \left(L - \frac{s}{k}M\right)^{-1} M \left(L - \frac{t-s}{l}M\right)^{-1} M(R_\beta^L(M))^2 v ds =$$

$$= \int\limits_0^t U_l^{t-s} U_k^s \left(\frac{s}{k} - \frac{t-s}{l}\right) \frac{k}{s} R^L_{\frac{k}{s}}(M) \frac{l}{t-s} R^L_{\frac{l}{t-s}}(M)((\beta L - M)^{-1}M)^2 v ds.$$

Taking into account (11), we get

$$\|U_k^t u - U_l^t u\|_{\mathcal{U}} \leq K^2 \int\limits_0^t \left|\frac{s}{k} - \frac{t-s}{l}\right| \frac{kl}{s(t-s)} \frac{K^2}{\frac{kl}{s(t-s)}} \|((\beta L - M)^{-1}M)^2 v\|_{\mathcal{U}} ds \leq$$

$$\leq \frac{K^4 t^2}{2} \left(\frac{1}{k} + \frac{1}{l}\right) \|((\beta L - M)^{-1}M)^2 v\|_{\mathcal{U}}. \qquad (15)$$

From (15), (10), (11) and density of $\operatorname{im} R_\mu^L(M)$ in $\mathcal{U}^1$ it follows the existence of the limit

$$\tilde{U}^t = s - \lim_{k\to\infty} U_k^t, \quad \tilde{U}^t \in \mathcal{L}(\tilde{\mathcal{U}}), \quad \|\tilde{U}^t\|_{\mathcal{L}(\tilde{\mathcal{U}})} \leq K \quad \forall t > 0.$$

Inequality (15) shows that $U_k^t u$ uniformly with respect to $t \in (0, T]$ converges to $\tilde{U}^t u$. Thus, the family $\{\tilde{U}^t : t > 0\}$ is strongly continuous with respect to t, because due to continuity of right L-resolvents of the operator it follows a strong continuity of the family $\{U_k^t : t > 0\}$ for any $k \in \mathbb{N}$. In order to extend the strong continuity of $\{\tilde{U}^t : t > 0\}$ up to zero, we define an element of the family of operators $\tilde{U}^0$ as a strong limit:

$$\tilde{U}^0 = s - \lim_{t\to 0+} \tilde{U}^t.$$

Due to (10)

$$\forall u \in \mathcal{U}^0 \quad \tilde{U}^0 u = \lim_{t\to 0+} \tilde{U}^t u = \lim_{t\to 0+} \lim_{k\to\infty} U_k^t u = 0$$

In addition, using the above-mentioned uniform convergence, it can be shown that

$$\forall u \in \mathcal{U}^1 \quad \tilde{U}^0 u = \lim_{t\to 0+} \tilde{U}^t u = \lim_{t\to 0+} \lim_{k\to\infty} U_k^t u = \lim_{k\to\infty} \lim_{t\to 0+} U_k^t u = u.$$

So, we get that $\tilde{U}^0 = \tilde{P}$.

Note that

$$U_{kl}^t = \frac{(-1)^{kl-1}}{(kl-1)!} \left(\frac{kl}{t}\right)^{kl} \left(\frac{d^{kl-1}}{d\mu^{kl-1}} R_\mu^L(M)\right)\Bigg|_{\mu=\frac{kl}{t}} =$$

$$= \frac{(-1)^{kl-2}}{(kl-2)!} \left(\frac{kl}{t}\right)^{kl} \left(\frac{d^{kl-2}}{d\mu^{kl-2}} \left(R_\mu^L(M)\right) R_\mu^L(M)\right)\Bigg|_{\mu=\frac{kl}{t}} = \ldots =$$

$$= \frac{(-1)^{l-1}}{(l-1)!} \left(\frac{kl}{t}\right)^{kl} \left(\frac{d^{l-1}}{d\mu^{l-1}} \left(R_\mu^L(M)\right) (R_\mu^L(M))^{(k-1)l}\right)\Bigg|_{\mu=\frac{kl}{t}} =$$

$$= \left(\left(\frac{l}{t/k}\right)^l \frac{(-1)^{l-1}}{(l-1)!} \frac{d^{l-1}}{d\mu^{l-1}} [R_\mu^L(M)]\right)^k \Bigg|_{\mu=\frac{kl}{t}} = \left(U_l^{\frac{t}{k}}\right)^k. \qquad (16)$$

From (11) it follows, that $\lim\limits_{l\to+\infty}\left(U_l^{\frac{t}{k}}\right)^k=\left(\tilde{U}^{\frac{t}{k}}\right)^k$. Indeed, at $u\in\tilde{\mathcal{U}}$

$$\left\|\left(U_l^{\frac{t}{k}}\right)^k u-\left(\tilde{U}^{\frac{t}{k}}\right)^k u\right\|_{\mathcal{U}}=\left\|\sum_{m=0}^{k-1}\left(U_l^{\frac{t}{k}}\right)^{k-m-1}\left(\tilde{U}^{\frac{t}{k}}\right)^m\left(U_l^{\frac{t}{k}}-\tilde{U}^{\frac{t}{k}}\right)u\right\|_{\mathcal{U}}\leq$$

$$\leq kK^{k-1}\left\|U_l^{\frac{t}{k}}u-\tilde{U}^{\frac{t}{k}}u\right\|_{\mathcal{U}}.$$

Tending in identity (16) $l\to+\infty$, we obtain

$$\tilde{U}^t=\left(\tilde{U}^{\frac{t}{k}}\right)^k. \tag{17}$$

Let us show that hence and from the strong continuity it follows that $\{\tilde{U}^t: t\geq 0\}$ is a semigroup. Taking rational $s=k/l$ и $t=m/n$ using (17) twice, we get

$$\tilde{U}^s\tilde{U}^t=\tilde{U}^{\frac{kn}{ln}}\tilde{U}^{\frac{lm}{ln}}=\left(\tilde{U}^{\frac{1}{ln}}\right)^{kn}\left(\tilde{U}^{\frac{1}{ln}}\right)^{lm}=\left(\tilde{U}^{\frac{1}{ln}}\right)^{kn+lm}=\tilde{U}^{\frac{kn+lm}{ln}}=\tilde{U}^{s+t}.$$

For arbitrary real numbers $s,t>0$ there exist sequences of rational numbers $\{s_n:n\in\mathbb{N}\},\{t_n:n\in\mathbb{N}\}$ such that $\lim\limits_{n\to\infty}s_n=s,\quad \lim\limits_{n\to\infty}t_n=t$. Then,

$$\forall n\in\mathbb{N}\quad \tilde{U}^{s_n}\tilde{U}^{t_n}=\tilde{U}^{s_n+t_n}. \tag{18}$$

Since,

$$\|\tilde{U}^{s_n}\tilde{U}^{t_n}u-\tilde{U}^s\tilde{U}^t u\|_{\mathcal{U}}\leq\|\tilde{U}^{s_n}\|_{\mathcal{L}(\tilde{\mathcal{U}})}\|\tilde{U}^{t_n}u-\tilde{U}^t u\|_{\mathcal{U}}+\|\tilde{U}^{s_n}\tilde{U}^t u-\tilde{U}^s\tilde{U}^t u\|_{\mathcal{U}}\leq$$

$$\leq K\|\tilde{U}^{t_n}u-\tilde{U}^t u\|_{\mathcal{U}}+\|\tilde{U}^{s_n}\tilde{U}^t u-\tilde{U}^s\tilde{U}^t u\|_{\mathcal{U}}, \tag{19}$$

tending $n\to\infty$ in (18), (19) and using the strong continuity with respect to t of the operators family $\{\tilde{U}^t:t\geq 0\}$, we obtain the desired.

Further, let $u^1=(R^L_\beta(M))^2v$ for some $\beta>0$ and $v\in\mathcal{U}$. Then

$$\|U_1^t u^1-\tilde{U}^t u^1\|_{\mathcal{U}}=\lim_{k\to\infty}\|U_1^t u^1-U_k^t u^1\|_{\mathcal{U}}\leq$$

$$\leq\lim_{k\to\infty}\frac{K^4t^2}{2}\left(1+\frac{1}{k}\right)\|((\beta L-M)^{-1}M)^2v\|_{\mathcal{U}}=\frac{K^4t^2}{2}\|((\beta L-M)^{-1}M)^2v\|_{\mathcal{U}},$$

due to (15).

$$U_1^t u^1=\left(\frac{1}{t}R^L_{\frac{1}{t}}(M)\right)u^1=\left(I+\left(\frac{1}{t}L-M\right)^{-1}M\right)u^1=$$

$$=u^1+R^L_{\frac{1}{t}}(M)R^L_\beta(M)(\beta L-M)^{-1}Mv=$$

$$=u^1+t(L-tM)^{-1}(L-tM+tM)R^L_\beta(M)(\beta L-M)^{-1}Mv=$$

$$=tR^L_\beta(M)(\beta L-M)^{-1}Mv+tR^L_{\frac{1}{t}}(M)R^L_\beta(M)((\beta L-M)^{-1}M)^2v.$$

Therefore

$$\left\|\frac{\tilde{U}^t-I}{t}u^1-R^L_\beta(M)(\beta L-M)^{-1}Mv\right\|_{\mathcal{U}}\leq$$

$$\leq \left\| \frac{U_1^t - I}{t} u^1 - R_\beta^L(M)(\beta L - M)^{-1} M v \right\|_{\mathcal{U}} + \left\| \frac{\tilde{U}^t - U_1^t}{t} u^1 \right\|_{\mathcal{U}} \leq$$

$$\leq tK \| ((\beta L - M)^{-1} M)^2 v \|_{\mathcal{U}} + \frac{K^4 t}{2} \| R_\beta^L(M)((\beta L - M)^{-1} M)^2 v \|_{\mathcal{U}}.$$

Tending $t \to 0+$ we obtain, that

$$\lim_{t \to 0+} \frac{\tilde{U}^t - I}{t} u^1 = R_\beta^L(M)(\beta L - M)^{-1} M v. \tag{20}$$

Act on (20) by the operator $\tilde{U}^s$ and get the differentiability on the right of the semigroup at this element $u^1 = (R_\beta^L(M))^2 v$ at point $s > 0$. In order to show the differentiability on the left at this point, one can consider the expression

$$\frac{\tilde{U}^{s-t} - \tilde{U}^s}{-t} u^1 = \frac{\tilde{U}^{s-t}(\tilde{U}^t - I)}{t} u^1, \quad s > t > 0,$$

proceed to the limit when $t \to 0+$, using the uniform boundedness of the semigroup. So, by virtue of (20)

$$\frac{d}{dt} \tilde{U}^t (R_\beta^L(M))^2 v = \tilde{U}^t R_\beta^L(M)(\beta L - M)^{-1} M v.$$

Act on last identity by the operator $R_\alpha^L(M)$.

By construction $\tilde{U}^t$ commutes with the operators $R_\alpha^L(M)$ and $(\alpha L - M)^{-1} M$ for the corresponding u^1, therefore by (20) we obtain

$$R_\alpha^L(M) \frac{d}{dt} \tilde{U}^t u^1 = (\alpha L - M)^{-1} M \tilde{U}^t u^1. \tag{21}$$

Obviously, for $u^0 \in \mathcal{U}^0$ $\tilde{U}^s(u^0 + u^1) = \tilde{U}^s u^1$. Then one can change u^1 by $u = u^0 + u^1 \in \mathcal{U}^0 \dot{+} \operatorname{im}(R_\mu^L(M))^2$ in identity (21). Thus, the function $u(t) = \tilde{U}^t u$ is the solution of the equation (7) for arbitrary u from the dense in $\tilde{\mathcal{U}}$ lineal $\mathcal{U}^0 \dot{+} \operatorname{im}(R_\mu^L(M))^2$.

(For the semigroup $\tilde{F}^t = s - \lim_{k \to \infty} F_k^t$, which is constructed by means of the left L-resolvent, the proof is identical).

□

The semigroup $\tilde{U}^t$ ($\tilde{F}^t$) at first is defined not on the whole space $\mathcal{U}$ ($\mathcal{F}$), but on some subspace $\tilde{\mathcal{U}}$ ($\tilde{\mathcal{F}}$). Introduce the sufficient condition of their coincidence: $\mathcal{U} = \tilde{\mathcal{U}}$ ($\mathcal{F} = \tilde{\mathcal{F}}$).

Theorem 3. [2] *Let the space $\mathcal{U}$ ($\mathcal{F}$) be reflexive, the operator M be weakly L-radial. Then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$ ($\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$).*

Definition 5. *Operator M is called strongly L-radial on the right (on the left), if it is L-radial and*

$$\| R_\mu^L(M)(\lambda L - M)^{-1} M u \|_{\mathcal{U}} \leq \frac{const(u)}{\lambda \mu} \quad \forall u \in \operatorname{dom} M$$

(there exists a dense in $\mathcal{F}$ lineal $\overset{\circ}{\mathcal{F}}$ such, that

$$\| M(\lambda L - M)^{-1} L_\mu^L(M) f \|_{\mathcal{F}} \leq \frac{const(f)}{\lambda \mu} \quad \forall f \in \overset{\circ}{\mathcal{F}})$$

for any $\lambda, \mu > 0$.

Theorem 4. [2] *Let the operator M be strongly L-radial on the right (on the left). Then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$ ($\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$).*

Remark 3. Obviously, that under the condition of strong L-radiality of operator M on the right (on the left) the resolving semigroup of the equation (7) ((8)) is defined on the whole space $\mathcal{U}$ $(\mathcal{F})$, and the projector $P \lim_{\mu\to+\infty} \mu R^L_\mu(M)$ $(Q = s - \lim_{\mu\to+\infty} \mu L^L_\mu(M))$ is it's unit.

References

1. Hille E., Phillips R.S. *Functional Analysis and Semi-Groups.* American Mathematical Society, Providence, Rhode Island, 1957.
2. Sviridyuk G.A., Fedorov V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators.* Utrecht, Boston, Köln, Tokyo, VSP, 2003.
3. Demidenko G.V., Uspenskii S.V. *Partial Differential Equations and Systems Not Solvable with Respect to the Highest Order Derivative.* N.Y., Basel, Hong Kong, Marcel Dekker, Inc., 2003.
4. Favini A., Yagi A. *Degenerate Differential Equations in Banach Spaces.* N.Y., Basel, Hong Kong, Marcel Dekker, Inc., 1999.
5. Sidorov N., Loginov B., Sinithyn A., Falaleev M. *Lyapunov-Shmidt Methods in Nonlinear Analysis and Applications.* Dordrecht, Boston, London, Kluwer Academic Publishers, 2002.
6. Al'shin A. B., Korpusov M.O., Sveshnikov A.G. *Blow-up in Nonlinear Sobolev Type Equations.* Series in Nonlinear Analisys and Applications, 15, De Gruyter, 2011.
7. Sviridyuk G.A. Linear Sobolev Type Equations and Strongly Continuous Semigroups of the Resolving Operators with Kernels [Lineynye uravneniya tipa Soboleva i sil'no nepreryvnye polugruppy razreshayushchikh operatorov s yadrami]. *Doklady akademii nauk,* 1994, vol. 337, no. 5, pp. 581–584.
8. Sviridyuk G.A., Zamyshlyaeva A.A. The Phase Spaces of a Class of Linear Higher-order Sobolev Type Equations. *Differential Equations*, 2006, vol. 42, no. 2. pp. 269–278.

УДК 517.9 + 519.216.2

СИЛЬНО НЕПРЕРЫВНЫЕ ПОЛУГРУППЫ ОПЕРАТОРОВ. АЛЬТЕРНАТИВНЫЙ ПОДХОД

А.А. Замышляева

Наследуя и продолжая традицию, восходящую к теореме Хилле–Иосиды–Феллера–Филлипса–Миядеры, в данной работе в рамках теории уравнений соболевского типа, которая переживает эпоху своего расцвета, рассмотрен новый способ построения аппроксимаций сильно непрерывных полугрупп операторов с ядрами. Вводится понятие относительно радиального оператора, содержащее условие в виде оценки производной относительной резольвенты, показывается существование C_0-полугруппы на некотором подпространстве исходного пространства, приводятся достаточные условия его совпадения со всем пространством. Результаты будут весьма полезными при численном исследовании многих неклассических математических моделей, рассматриваемых в рамках теории уравнений соболевского типа первого порядка, а также для распространения идей и методов на уравнения соболевского типа высокого порядка.

Ключевые слова: уравнение соболевского типа, сильно непрерывные полугруппы операторов с ядрами, аппроксимации полугруппы.

Литература

1. Хилле, Э. Функциональный анализ и полугруппы / Э. Хилле, Р. Филлипс. – М.: ИЛ, 1962.
2. Sviridyuk, G.A. Linear Sobolev Type Equations and Degenerate Semigroups of Operators / G.A. Sviridyuk, V.E. Fedorov. – Utrecht; Boston; Köln; Tokyo: VSP, 2003.
3. Demidenko, G.V. Partial Differential Equations and Systems not Solvable with Respect to the Highest Order Derivative / G.V. Demidenko, S.V. Uspenskii. – N.Y.; Basel;Hong Kong: Marcel Dekker, Inc., 2003.
4. Favini, A. Degenerate Differential Equations in Banach Spaces / A. Favini, A. Yagi. – N.Y.; Basel; Hong Kong: Marcel Dekker, Inc., 1999.
5. Lyapunov-Shmidt Methods in Nonlinear Analysis and Applications / N. Sidorov, B. Loginov, A. Sinithyn, M. Falaleev. – Dordrecht; Boston; London: Kluwer Academic Publishers, 2002.
6. Al'shin, A.B. Blow-up in Nonlinear Sobolev Type Equations /A.B. Al'shin, M.O. Korpusov, A.G. Sveshnikov. – Series in nonlinear analisys and applications, 15, De Gruyter, 2011.
7. Свиридюк, Г.А. Линейные уравнения типа Соболева и сильно непрерывные полугруппы разрешающих операторов с ядрами / Г.А. Свиридюк // ДАН. – 1994. – Т. 337, №5. – С. 581–584.
8. Sviridyuk, G.A. The Phase Spaces of a Class of Linear Higher-order Sobolev Type Equations / G.A. Sviridyuk, A.A. Zamyshlyaeva // Differential Equations. – 2006. – V. 42, №2. – P. 269–278.

Алена Александровна Замышляева, кандидат физико-математических наук, доцент, кафедра «Уравнения математической физики», Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), alzama@mail.ru.

Поступила в редакцию 4 марта 2013 г.