

## THE LYAPUNOV STABILITY OF THE CAUCHY–DIRICHLET PROBLEM FOR THE GENERALIZED HOFF EQUATION

*P.O. Moskvicheva*, South Ural State University, Chelyabinsk, Russian Federation,  
pelageia@bk.ru,

*I.N. Semenova*, Ural State Pedagogical University, Yekaterinburg, Russian Federation,  
semenova\_i\_n@mail.ru

We consider the initial boundary value problem with homogeneous Dirichlet boundary conditions for the generalized Hoff equation in a bounded domain. This equation models the dynamics of buckling of a double-tee girder under constant load and belongs to a large class of Sobolev type semilinear equations (We can isolate the linear and non-linear parts of the operator acting on the original function). The paper addresses the stability of zero solution of this problem. There are two methods in the theory of stability: the first one is the study of stability by linear approximation and the second one is the study of stability by Lyapunov function. We use the second Lyapunov's method adapted to the case of incomplete normed spaces. The main result of this paper is a theorem on the stability and asymptotic stability of zero solution to this problem.

*Keywords:* Sobolev-type equation; phase space; Lyapunov stability.

### Introduction

Let  $\Omega \subset \mathbb{R}^s$  be a bounded domain with boundary  $\partial\Omega$  of class  $C^\infty$ . Consider the generalized Hoff equation [1] in cylinder  $\Omega \times \mathbb{R}$

$$(\lambda - \lambda_0)u_t + \Delta u_t = \alpha_1 u + \alpha_2 u^3 + \dots + \alpha_n u^{2n-1}, \quad n \in \mathbb{N}. \quad (1)$$

This equation models the bending of an I-beam. Here the function  $u = u(x, t)$ ,  $(x, t) \in \Omega \times \mathbb{R}$  is the displacement of the beam from the vertical position. The parameter  $\lambda \in \mathbb{R}_+$  corresponds to a constant vertical load and the parameters  $\alpha_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$  characterize the material of the beam.

Consider the initial-boundary value problem

$$u(x, 0) = u_0(x), x \in \Omega; \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R} \quad (2)$$

for equation (1). This problem was firstly considered in [2 – 4], wherein it was found out that the problem is essentially unsolvable for arbitrary initial data. The set of initial values, which guarantees the existence and uniqueness of solution to initial-boundary value problem for equation (1), has been studied in [5]. If  $n = 2$  and  $\alpha_1 \cdot \alpha_2 \in \mathbb{R}_+$  then the phase space of equation (1) is a simple Banach  $C^\infty$ -manifold. This result was obtained in [6]. And if  $\alpha_1 \cdot \alpha_2 \in \mathbb{R}_-$  then the phase space of equation (1) lies on the Whitney fold. It is shown in [7]. The generalized Hoff equation (for  $n > 3$ ) was considered in [8], but in this paper the stability has not been studied. This result was obtained in [9] for the case when  $n = 3$ . Generalized Sobolev type equations have been studied in other papers, for example, in [10]. In this paper we generalize the results of [9] and consider the case when  $n > 3$ .

The paper consists of two parts. The first part is devoted to the reduction of problem (1), (2) to the Cauchy problem

$$u(0) = u_0 \tag{3}$$

for abstract semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u). \tag{4}$$

Here  $L, M$  are linear operators and  $N$  is nonlinear operator defined on specially constructed functional spaces. The second part is devoted to the study of stability of stationary solution to problem (1), (2). This is the main result.

## 1. Phase Space

Consider spaces  $\mathfrak{U} = \overset{\circ}{W}_2^1(\Omega)$ ,  $\mathfrak{F} = W_2^{-1}(\Omega)$  and operators

$$\langle Lu, v \rangle = \int_{\Omega} ((\lambda - \lambda_0)uv - \nabla u \nabla v) dx, \quad \forall u, v \in \overset{\circ}{W}_2^1(\Omega), \tag{5}$$

$$\langle Mu, v \rangle = \alpha_1 \int_{\Omega} uv dx, \quad \forall u, v \in L_{2n}(\Omega), \tag{6}$$

$$\langle N(u), v \rangle = \int_{\Omega} (\alpha_2 u^3 + \dots + \alpha_{n-1} u^{2n-3} + \alpha_n u^{2n-1}) v dx \quad \forall u, v \in L_{2n}(\Omega). \tag{7}$$

Embedding  $\overset{\circ}{W}_2^1(\Omega) \hookrightarrow L_{2n}(\Omega)$  is dense and continuous. Therefore  $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ , and  $L$  is a Fredholm operator. The spectrum of the operator  $L$  is real and discrete, has finite multiplicity and condenses only to  $-\infty$ . Operator  $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$ .

A vector function  $u \in C^\infty(\mathfrak{U}, \mathfrak{F})$  satisfying equation (4) is called a solution of this equation.

**Definition 1.** A set  $\mathfrak{P} \subset \mathfrak{U}$  is called a *phase space* of equation (3) if the following conditions are satisfied:

- (i) each solution  $u = u(t)$  of (4) lies in  $\mathfrak{P}$  pointwisely; i.e.,  $u(t) \in \mathfrak{P}$  for all  $t \in \mathbb{R}$ ;
- (ii) for each  $u_0 \in \mathfrak{P}$ , there exists a unique solution to problem (3), (4).

**Theorem 1.** Let  $n = 1, 2$  when  $s = 4$  (i.e.  $\Omega \subset \mathbb{R}^4$ ),  $n = 1, 2, 3$  when  $s = 3$  (i.e.  $\Omega \subset \mathbb{R}^3$ ) and  $n \in \mathbb{N}$  when  $s = 1, 2$  (i.e.  $\Omega \subset \mathbb{R}$  or  $\Omega \subset \mathbb{R}^2$ ). Then one of the two conditions is satisfied

- (i) if  $\ker L = \{0\}$ , then the phase space of equation (1) coincides with  $\mathfrak{U}$ .
- (ii) if  $\ker L \neq \{0\}$ , and all coefficients  $\alpha_i \in \mathbb{R} \setminus \{0\}$ ,  $i = 1, \dots, n$  have the same sign. Then the phase space of equation (1) is simple manifold

$$\mathfrak{M} = \left\{ u \in \mathfrak{U} : \int_{\Omega} (\alpha_1 + \alpha_2 u^2 + \dots + \alpha_n u^{2n-2}) u \chi_k dx = 0, \quad k = 1, \dots, m \right\}.$$

Here  $\chi_k$  are orthonormal eigenfunctions corresponding to the eigenvalues  $\lambda_k$  of  $L$ .

## 2. Stability

**Definition 2.** A family of mappings  $S$  is called a *nonlinear semigroup* in a normed space  $\mathfrak{V}$  if for every  $u \in \mathfrak{V}$  and some  $\tau = \tau(u) \in \mathbb{R}_+$  the following conditions are satisfied:

- (i)  $S = S(t, u) \in \mathfrak{V}$ , for all  $t \in (-\tau; \tau)$ ;  $S(0, u) = u$ ;
- (ii)  $S(t + s, u) = S(t, S(s, u))$  for all  $t + s \in (-\tau, \tau)$ .

A point  $u \in \mathfrak{V}$  such that  $S(t, u) = u, t \in \mathbb{R}$  is called a *stationary point*.

**Definition 3.** A stationary point  $u$  is called

- (i) *stable* (in sense of A.M. Lyapunov), if for any neighborhood  $\mathfrak{D}_u$  of  $u$  there exists a neighborhood  $\mathfrak{D}'_u$  (i.e. not necessarily the same neighborhood) of the same point, such that  $S(t, v) \in \mathfrak{D}_u$  for all  $v \in \mathfrak{D}'_u$  and  $t \in \mathbb{R}_+$ ;
- (ii) *asymptotically stable* (in sense of A.M. Lyapunov), if it is stable and for any point  $v$  in some neighborhood  $\mathfrak{D}_u$  of  $u$   $S(t, v) \rightarrow u$  for  $t \rightarrow \infty$ .

**Definition 4.** A functional  $V \in C(\mathfrak{V}; \mathbb{R})$  is called a *Lyapunov functional* if

$$\dot{V}(u) = \overline{\lim}_{t \rightarrow 0^+} \frac{1}{t} (V(S(t, u)) - V(u)) \leq 0$$

for all  $u \in \mathfrak{V}$ .

**Theorem 2.** Let  $u$  be a stationary point. If there exists a Lyapunov functional such that

- (i)  $V(u) = 0$ ;
- (ii)  $V(v) \geq \varphi(\|v - u\|)$ ; here  $\varphi$  is strictly increasing continuous function such that  $\varphi(0) = 0$  and  $\varphi(r) > 0$  for  $r \in \mathbb{R}_+$ , then the point  $u$  is stable.

**Theorem 3.** Let the conditions of Theorem 2 be satisfied, and a strictly increasing continuous function  $\psi$ , such that  $\psi(0) = 0$  and  $\psi(r) > 0$  for  $r \in \mathbb{R}_+$ , exist. If  $\dot{V}(v) \leq -\psi(\|v - u\|)$ , then the point  $u$  is asymptotically stable.

We will study the stability of problem (1),(2) using Theorems 2 and 3.

Consider a space  $\mathfrak{U}(=\overset{\circ}{W}_2^1)$  with norm  $\|\cdot\|$  of space  $L_2$ . It is an incomplete normed space. Define the Lyapunov functional by formula

$$V(u) = \int_{\Omega} (u_x^2 + (\lambda_0 - \lambda)u^2) dx.$$

Obviously  $V(0) = 0$  and  $V(u) \geq c\|u\|^2$ . Moreover multiplying (1) scalarly in  $L_2$  by  $u$  we obtain

$$\dot{V}(u) = -\alpha_1\|u\|^2 - \alpha_2\|u\|_{L_4}^4 - \dots - \alpha_n\|u\|_{L_{2n}}^{2n}.$$

Since the embedding  $L_{2n} \hookrightarrow L_2$  is obvious, the following inequality holds

$$\dot{V}(u) \leq -\alpha_1\|u\|^2 - \alpha_2c_2\|u\|_{L_4}^4 - \dots - \alpha_nc_n\|u\|_{L_{2n}}^{2n}. \quad (8)$$

Here  $c_j, j = \overline{1, n}$  are the constants of embedding. Function constructed from the norm  $\|\cdot\|$  on the right side of (8) satisfies the conditions of Theorem 3. So we have proved the following theorem.

**Theorem 4.** *Zero solution of problem (1), (2) is asymptotically stable for any  $\alpha_j \in \mathbb{R}_+, j = \overline{1, n}, \lambda \in [0, \lambda_0]$ .*

## References

1. Hoff N.J. Creep Buckling. *Aeronautical Quarterly*, 1956, vol. 7, no. 1, pp. 1–20.
2. Sidorov N.A. *Obshhie voprosy reguljarizacii v zadachah teorii vetvlenija* [Common Questions of Regularity in Problems of Ramification Theory]. Irkutsk, Irkutsk Gos. Univ. Publ., 1982. 314 p. (in Russian)
3. Sidorov N.A., Romanova O.A. [Application of Certain Results of Branching Theory in the Solution of Degenerate Differential Equations.] *Differetial'niye Uravneniya* [Differential Equations], 1983, vol. 19, no. 9, pp. 1516–1526. (in Russian)
4. Sidorov N.A., Falaleev M.V. [Generalized Solutions of Differential Equations with a Fredholm Operator at the Derivative]. *Differetial'niye Uravneniya* [Differential Equations], 1987, vol. 23, no. 4, pp. 726–728. (in Russian)
5. Sviridyuk G.A. Quasistationary Trajectories of Semilinear Dynamical Equations of Sobolev Type. *Russian Academy of Sciences. Izvestiya Mathematics*, 1994, vol. 42, no. 3, pp. 601–614. DOI: 10.1070/IM1994v042n03ABEH001547
6. Sviridyuk G.A., Kazak V.O. The Phase Space of an Initial-Boundary Value Problem for the Hoff Equation. *Mathematical Notes*, 2002, vol. 71, no. 2, pp. 262–266. DOI: 10.1023/A:1013919500605
7. Sviridyuk G.A., Trineeva I.K. A Whitney Fold in the Phase Space of the Hoff Equation. *Russian Mathematics*, 2005, vol. 49, no. 10, pp. 49–55.
8. Bajazitova A.A. [The Phase Space of the Initial-Boundary Value Problem for a Generalized Hoff Equation]. *Vestnik MaGU. Matematika* [Bulletin of the Magnitogorsk State University], 2010, vol. 12, pp. 15–21. (in Russian)
9. Zagrebina S.A., Pivovarova P.O. [Stability and Instability of the Solutions of the Hoff Equations. The Numerical experiment]. *Nonclassical Equations of Mathematical Physics*, Novosibirsk, Izdatel'stvo Instituta Matematiki Im. S.L. Soboleva SO RAN, 2010, pp. 88–94. (in Russian)
10. Sviridyuk G.A., Semenova I.N. Solvability of an Inhomogeneous Problem for a Generalized Boussinesq Filtration Equation. *Differential Equations*, 1988, vol. 24, no. 9, pp. 1065–1069.

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## УСТОЙЧИВОСТЬ ПО ЛЯПУНОВУ ЗАДАЧИ КОШИ – ДИРИХЛЕ ДЛЯ ОБОБЩЕННОГО УРАВНЕНИЯ ХОФФА

*П.О. Москвичева, И.Н. Семенова*

В данной статье исследуется начально-краевая задача Коши с однородными граничными условиями Дирихле для обобщенного уравнения Хоффа, заданного в ограниченной области. Это уравнение моделирует динамику выпучивания двутавровой балки, находящейся под постоянной нагрузкой и относится к классу полулинейных (у оператора действующего на исходную функцию можно выделить линейную часть и нелинейную) уравнений соболевского типа. Нас интересует устойчивость нулевого решения данной задачи. В рамках теории устойчивости выделяют два метода: первый — исследование устойчивости по линейному приближению и второй — исследование устойчивости посредством функции Ляпунова. Отметим, что первым методом Ляпунова исследовать устойчивость решения уравнения Хоффа, заданного в области, не удастся, поскольку в нашем случае относительный спектр оператора  $M$  пересекается с мнимой осью. Поэтому для нашей задачи был применен метод функций Ляпунова, модифицированный для случая неполных нормированных пространств. В результате получена теорема об устойчивости и асимптотической устойчивости нулевого решения данной задачи.

*Ключевые слова:* уравнение соболевского типа; фазовое пространство; устойчивость по Ляпунову.

### Литература

1. Hoff, N.J. Creep buckling / N.J. Hoff // *Aeronautical Quarterly*. – 1956. – V. 7, № 1. – P. 1–20.
2. Сидоров, Н.А. Общие вопросы регуляризации в задачах теории ветвления / Н.А. Сидоров. – Иркутск: Изд-во Иркутского гос. ун-та, 1982. – 314 с.
3. Сидоров, Н.А. О применении некоторых результатов теории ветвлений при решении дифференциальных уравнений / Н.А. Сидоров, О.А. Романова // *Дифференциальные уравнения*. – 1983. – Т. 19, № 9. – С. 1516–1526.
4. Сидоров, Н.А. Обобщенные решения дифференциальных уравнений с фредгольмовым оператором при производной / Н.А. Сидоров, М.В. Фалалеев // *Дифференциальные уравнения*. – 1987. – Т. 23, № 4. – С. 726–728.
5. Свиридюк, Г.А. Квазистационарные траектории полулинейных динамических уравнений типа Соболева / Г.А. Свиридюк // *Изв. РАН. Серия математическая*. – 1993. – Т. 57, № 3. – С. 192–207.
6. Свиридюк, Г.А. Фазовое пространство начально-краевой задачи для уравнения Хоффа / Г.А. Свиридюк, В.О. Казак // *Математические заметки*. – 2002. – Т. 71, № 2. – С. 292–297.
7. Свиридюк, Г.А. Сборка Уитни в фазовом пространстве уравнения Хоффа / Г.А. Свиридюк, И.К. Тринеева // *Известия вузов. Математика*. – 2005. – № 10. – С. 54–60.

8. Баязитова, А.А. Фазовое пространство начально-краевой задачи для обобщенного уравнения Хоффа / А.А. Баязитова // Вестник МаГУ. Математика. – 2010. – Вып. 12. – С. 15–21.
9. Загребина, С.А. Устойчивость и неустойчивость решений уравнений Хоффа. Численный эксперимент / С.А. Загребина, П.О. Пивоварова // Неклассические уравнения математической физики: сб. науч. работ / под ред. А.И. Кожанова. – Новосибирск, 2010. – С. 88–94.
10. Свиридюк, Г.А. Разрешимость неоднородной задачи для обобщенного фильтрационного уравнения Буссинеска / Г.А. Свиридюк, И.Н. Семенова // Дифференциальные уравнения. – 1988. – Т. 24, № 9. – С. 1607–1611.

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Полина Олеговна Москвичева, кандидат физико-математических наук, кафедра «Уравнения математической физики», Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), [relageia@bk.ru](mailto:relageia@bk.ru).

Ирина Николаевна Семенова, кандидат педагогических наук, кафедра «Теория и методика обучения математике», Уральский государственный педагогический университет (г. Екатеринбург, Российская Федерация), [semenova\\_i@mail.ru](mailto:semenova_i@mail.ru).

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