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ON A CLASS OF SOBOLEV-TYPE EQUATIONS

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The article surveys the works of T.G. Sukacheva and her students studying the models of incompressible viscoelastic Kelvin–Voigt fluids in the framework of the theory of semilinear Sobolev-type equations. We focus on the unstable case because of greater generality. The idea is illustrated by an example: the non-stationary thermoconvection problem for the order 0 Oskolkov model. Firstly, we study the abstract Cauchy problem for a semilinear nonautonomous Sobolev-type equation. Then, we treat the corresponding initial-boundary value problem as its concrete realization. We prove the existence and uniqueness of a solution to the stated problem. The solution itself is a quasi-stationary semi-trajectory. We describe the extended phase space of the problem. Other problems of hydrodynamics can also be investigated in this way: for instance, the linearized Oskolkov model, Taylor's problem, as well as some models describing the motion of an incompressible viscoelastic Kelvin–Voigt fluid in the magnetic field of the Earth.

Keywords: Sobolev type equations; incompressible viscoelastic fluids; relatively p-sectorial operators; extended phase spaces.

Introduction

The system

$$\begin{cases} (1-\lambda\nabla^2)\mathbf{v}_t = \nu\nabla^2\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p - g\gamma\theta + \mathbf{f}, \\ 0 = \nabla\cdot\mathbf{v}, \\ \theta_t = æ\nabla^2\theta - \mathbf{v}\cdot\nabla\theta + \mathbf{v}\cdot\gamma \end{cases} \tag{1}$$

models the evolution of the velocity $\mathbf{v} = (v_1,\ v_2,\ \ldots,\ v_n)$, $\quad v_i = v_i(x,t)$, pressure gradient $\nabla p = (p_1,\ p_2,\ \ldots,\ p_n)$, $\quad p_i = p_i(x,t)$ and temperature $\theta = \theta(x,t)$ of the simplest non Newton fluid — incompressible viscoelastic Kelvin – Voight fluid [1, 2].

The parameters $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R}_+$ and $æ \in \mathbb{R}_+$ characterize elasticity, viscosity and heat conduction of the fluid respectively; $g \in \mathbb{R}_+$ is the acceleration of the gravity; $\gamma = (0,\ \ldots,\ 0,\ 1)$ is the unit vector in $\mathbb{R}^n$; the free term $\mathbf{f} = (f_1,\ \ldots,\ f_n)$, $\quad f_i = f_i(x,t)$, corresponds to the external influence on the fluid.

We investigate the solvability of the initial-boundary value problem

$$\begin{aligned} &\mathbf{v}(x,0) = \mathbf{v}_0(x), \qquad \theta(x,0) = \theta_0(x), \qquad \forall x \in \Omega; \\ &\mathbf{v}(x,t) = 0, \qquad \theta(x,t) = 0, \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}_+ \end{aligned} \tag{2}$$

for system (1). Here $\Omega \subset \mathbb{R}^n$, $n = 2, 3, 4$, is a bounded domain with a smooth boundary $\partial\Omega$ of class $\mathcal{C}^\infty$. A.P. Oskolkov [3, 4] began to study problem (1), (2) and investigated the solvability of this problem in case $\lambda^{-1} > -\lambda_1$ (λ_1 is the least eigenvalue of Laplace operator with homogeneous Direchlet condition in the domain Ω).

The first initial-boundary value problem (2) for system (1) was considered by G.A. Sviridyuk [5, 6], and its modification for the flat parallel current was studied by him in [7]. In these papers the indicated problem was studied when the free term $\mathbf{f}$ did not depend on time under other assumptions (less general in Sviridyuk's papers [6, 7]) on the corresponding differential operators.

Our aim is to study the solvability of problem (1), (2) when the free term $\mathbf{f} = \mathbf{f}(x,t)$ is not stationary. We consider this problem in the frame of the Sobolev type equations theory. The base of this theory was created by professor Sviridyuk and now this theory is actively developed by his followers. This problem is investigated on the base of the concept of relatively p-sectorial operators and degenerative semi-groups of operators. The existence theorem of unique solution to this problem is proved and the description of its extended phase space is obtained.

So at first we study the abstract Cauchy problem and then consider problem (1), (2) as its concrete interpretation. We prove the existence theorem of the unique local solution to problem (1), (2). This solution is a quasi-stationary semi-trajectory.

Other models of incompressible viscoelastic fluids may be studied in the same way as problem (1), (2). The examples of these models will be indicated at the end of the paper.

1. Abstract Cauchy Problem for the Semi-Linear Non-Autonomous Sobolev Type Equations

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, i.e. it is linear and continuous, $\ker L \neq \{0\}$; operator $M : \operatorname{dom} M \to \mathcal{F}$ is linear and closed and it is densely defined in $\mathcal{U}$, i.e. $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$. Denote by $\mathcal{U}_M$ be the lineal $\operatorname{dom} M$, endowed with the norm of graph $|||\cdot||| = \|M\cdot\|_{\mathcal{F}} + \|\cdot\|_{\mathcal{U}}$. Assume that $F \in \mathcal{C}^\infty(\mathcal{U}_M;\mathcal{F})$, and function $f \in \mathcal{C}^\infty(\bar{\mathbb{R}}_+;\mathcal{F})$.

Consider the Cauchy problem

$$u(0) = u_0 \tag{3}$$

for the *semilinear non-autonomous Sobolev type equation*

$$L\,\dot{u} \;=\; M\,u \;+\; F(u) \;+ f(t). \tag{4}$$

Definition 1. *A vector-function $u \in \mathcal{C}^\infty((0,T);\mathcal{U}_M)$ is a solution to problem (3), (4) if it satisfies (3), (4) so that $u(t) \to u_0$ when $t \to 0+$.*

Let's introduce the L-resolving set

$$\rho^L(M) \;=\; \{\mu \in \mathbb{C} :\; (\mu L - M)^{-1} \;\in \mathcal{L}(\mathcal{F};\mathcal{U})\}$$

and the L-spectrum $\sigma^L(M) \;=\; \mathbb{C} \setminus \rho^L(M)$ of the operator M.

Definition 2. *An operator M is an (L,p)-sectorial one, if there exist the constants $a \in \mathbb{R},\ k \in \mathbb{R}_+,\ \Theta \in (\frac{\pi}{2}, \pi)$ such that*

(i) $S^L_{\Theta,a}(M) = \{\mu \in \mathbb{C} :\ |\arg(\mu - a)| < \Theta\ ,\ \ \mu \neq a\ \} \subset \rho^L(M)$;

(ii) $\max\{\ \| R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathcal{U})},\ \ \ \| L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathcal{F})}\ \} \ \leq\ \dfrac{k}{\prod_{q=0}^{p}|\mu_q - a|}$

for all $\mu,\ \mu_0,\ \mu_1,\ \dots\ ,\mu_p \in S^L_{\Theta,a}(M)$.

Here

$$R^L_{(\mu,p)}(M) = \Pi^p_{q=0} R^L_{\mu_q}(M), \qquad L^L_{(\mu,p)}(M) = \Pi^p_{q=0} L^L_{\mu_q}(M)$$

are right and left (L,p)-resolvents of operator M respectively, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ [10].

Definition 3. *An operator M is a strongly (L,p)-sectorial one, if it is (L,p)-sectorial and for all μ, $\mu_0, \dots, \mu_p \in S^L_{\Theta,a}(M)$*

$$\text{(i) } \| M R^L_{(\mu,p)}(M)\,(\,\mu L - M\,)^{-1} f \|_{\mathcal{F}} \le \frac{\text{const}(f)}{|\mu - a| \prod_{q=0}^{p} |\mu_q - a|}$$

for all f from some dense in $\mathcal{F}$ lineal;

$$\text{(ii) } \| \,(\,\mu L - M\,)^{-1} L^L_{(\mu,p)}(M)\, \|_{\mathcal{L}(\mathcal{F};\mathcal{U})} \le \frac{\text{const}}{|\mu - a| \prod_{q=0}^{p} |\mu_q - a|}.$$

Remark 1. If $p = 0$, then (L,p)-sectorial and strongly (L,p)-sectorial operator M is called L-sectorial and strongly L-sectorial [5] respectively.

Consider problem (3), (4) and suppose that operator M is strongly (L,p)-sectorial. Under this assumption the solution to this problem may be non-unique. Consider the following example.

Example 1. [11] Let $\mathcal{U}_M = \mathcal{U} = \mathcal{F} = \mathbb{R}^2$, and operators L, M and F be given by

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F(x) = \begin{pmatrix} 0 \\ -x_1^2 \end{pmatrix}.$$

The operator M is strongly $(L,1)$-sectorial, since

$$(\mu L - M)^{-1} = \begin{pmatrix} -1 & -\mu \\ 0 & -1 \end{pmatrix}, \quad R^L_\mu(M) = L^L_\mu(M) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Consider the Cauchy problem $x(0) = 0$, where $x = \text{col}\,(x_1, x_2)$ for the equation

$$L\,\dot{x} \;=\; M\,x \;+\; F(x).$$

There exist two solutions to this problem col $(0,0)$ and col $(t/2, t^2/4)$.

Thus the solution of problem (3), (4) may be non-unique. So we restrict the idea of the solution to equation (4). Moreover it is known [12–14], that solutions to problem (3), (4) exist not for all $u_0 \in \mathcal{U}_M$. Thus we introduce two definitions.

Definition 4. *[15]. The set $\mathcal{B}^t \subset \mathcal{U}_M \times \bar{\mathbb{R}}_+$ is called an extended phase space of equation (4), if for every point $u_0 \in \mathcal{U}_M$ such that $(u_0, 0) \in \mathcal{B}^0$ there exists a unique solution to problem (3), (4), and $(u(t), t) \in \mathcal{B}^t$.*

Remark 2. If $\mathcal{B}^t = \mathcal{B} \times \bar{\mathbb{R}}_+$, where $\mathcal{B} \subset \mathcal{U}_M$, then the set $\mathcal{B}$ is called a *phase space* of equation (4).

Definition 5. *Let the space* $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1$ *and* $\ker L \subset \mathcal{U}_0$. *The solution* $u = v + w$, *where* $v(t) \in \mathcal{U}_0$, *and* $w(t) \in \mathcal{U}_1$ *for all* $t \in (0, T)$, *to equation (4) is called a quasi-stationary semi-trajectory, if* $L\dot{v} \equiv 0$.

Remark 3. The concept of *quasi-stationary semi-trajectory* generalizes the concept of *quasi-stationary trajectory*, introduced for the dynamic case [11, 13, 14].

It is well known that if the operator M is strongly (L, p)-sectorial, then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$, where

$$\mathcal{U}^0 = \{\varphi \in \mathcal{U} : U^t\varphi = 0 \ \ \exists t \in \mathbb{R}_+\}, \quad \mathcal{F}^0 = \{\psi \in \mathcal{F} : F^t\psi = 0 \ \ \exists t \in \mathbb{R}_+\}$$

are *kernels*, and

$$\mathcal{U}^1 = \{u \in \mathcal{U} : \lim_{t\to 0+} U^t u = u\}, \quad \mathcal{F}^1 = \{f \in \mathcal{F} : \lim_{t\to 0+} F^t f = f\}$$

are *images* of the analytic resolving semigroups

$$U^t = \frac{1}{2\pi i}\int_\Gamma R^L_\mu(M)e^{\mu t}d\mu, \qquad F^t = \frac{1}{2\pi i}\int_\Gamma L^L_\mu(M)e^{\mu t}d\mu, \tag{5}$$

($\Gamma \subset S^L_{\Theta,a}(M)$ is a contour such that $\arg\mu \to \pm\Theta$ when $|\mu| \to +\infty$) of the linear homogeneous equation $L\dot{u} = Mu$. Let $L_k(M_k)$ be the restriction of the operator $L(M)$ on $\mathcal{U}^k$ $(\mathcal{U}^k \cap \operatorname{dom} M)$, $k = 0,\ 1$. Then $L_k : \mathcal{U}^k \to \mathcal{F}^k$, $M_k : \mathcal{U}^k \cap \operatorname{dom} M \to \mathcal{F}^k$, $k = 0,\ 1$; and restrictions M_0 and L_1 of operators M and L on the spaces $\mathcal{U}^0 \cap \operatorname{dom} M$ and $\mathcal{U}^1$ respectively are linear continuous operators and they have bounded reverse operators. [10].

So we reduce equation (4) to the equivalent form

$$\begin{aligned} R\dot{u}^0 &= u^0 + G(u) + g(t) \qquad u^0(0) = u^0_0, \\ \dot{u}^1 &= Su^1 + H(u) + h(t) \qquad u^1(0) = u^1_0, \end{aligned} \tag{6}$$

where $u^k \in \mathcal{U}^k$, $k = 0, 1$, $u = u^0 + u^1$, operators $R = M_0^{-1}L_0$, $S = L_1^{-1}M_1$, $G = M_0^{-1}(I - Q)F$, $H = L_1^{-1}QF$, $g = M_0^{-1}(I - Q)f$, $h = L_1^{-1}Qf$. Here $Q \in \mathcal{L}(F)(\equiv \mathcal{L}(F; F))$ is the corresponding projector.

Definition 6. *System of equations (6) is called a normal form of equation (4).*

Remark 4. In the case, when operator M is strongly L-sectorial, the normal form of the equation (4) $(f(t) \equiv 0)$ coincides with the form in [7].

We study quasi-stationary semi-trajectories of equation (4), for which $R\dot{u}^0 \equiv 0$. So we assume that the operator R is *bi-splitting* [16], i.e. its kernel $\ker R$ and image $\operatorname{im} R$ are completable in space $\mathcal{U}$. Denote $\mathcal{U}^{00} = \ker R$, and $\mathcal{U}^{01} = \mathcal{U}^0 \ominus \mathcal{U}^{00}$ is a complement of the subspace $\mathcal{U}^{00}$. Then the first equation of the normal form (6) is reduced to the form

$$R\dot{u}^{01} = u^{00} + u^{01} + G(u) + g(t), \tag{7}$$

where $u = u^{00} + u^{01} + u^1$.

Theorem 1. *Let operator* M *be strongly* (L, p)*-sectorial, and operator* R *be bi-splitting. If there exists a quasi-stationary semi-trajectory* $u = u(t)$ *of equation (4), then it satisfies the following relations*

$$0 = u^{00} + u^{01} + G(u) + g(t), \quad u^{01} = \text{const}. \tag{8}$$

Proof. The first relation follows from (7) due to the requirement of quasisteadyness $R\dot{u}^0 = R\dot{u}^{01} \equiv 0$. The second relation follows from the identity $R\dot{u}^{01} \equiv 0$, because in the Banach theorem about the inverse operator the restriction of operator $Q_R R(I - P_R)$ on $\mathcal{U}^{01}$ is a continuously invertible operator. Here Q_R and P_R are the projectors on $\operatorname{im} R$ and $\ker R$ respectively, $\ker P_R = \mathcal{U}^{01}$.

□

Remark 5. The second relation in (8) explains the meaning of the term *quasi-stationary semi-trajectories*, i.e. such semi-trajectories, which are stationary on some variables. In the other words the quasi-stationary semi-trajectory necessarily lies in the plane $(I - P_R)u^0 = \text{const}$.

Theorem 1 establishes the necessary condition for the existence of quasi-stationary semi-trajectory of equation (4). Now consider the sufficient conditions.

It is known that if the operator M is strongly (L,p)-sectorial then the operator S is sectorial [10]. So it generates an analytic semigroup on $\mathcal{U}^1$. Denote it by $\{U_1^t : t \geq 0\}$ since the operator U_1^t in fact is the restriction of the operator U^t on $\mathcal{U}^1$. The fact that $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$ shows that there exists a projector $P \in \mathcal{L}(\mathcal{U})$, corresponding to this splitting. It can be shown that $P \in \mathcal{L}(\mathcal{U}_M)$ [10]. Then the space $\mathcal{U}_M$ splits into the direct sum $\mathcal{U}_M = \mathcal{U}_M^0 \oplus \mathcal{U}_M^1$ so that the embedding $\mathcal{U}_M^k \subset \mathcal{U}^k, \quad k = 0,\ 1$, is dense and continuous. Symbol A'_v denotes the Frechet derivative of an operator A, defined on some Banach space $\mathcal{V}$, at $v \in \mathcal{V}$.

Theorem 2. *Let operator M be strongly (L,p)-sectorial, operator R be bi-splitting, operator $F \in \mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$, vector-function $f \in \mathcal{C}^\infty(\mathbb{R}_+; \mathcal{F})$, and the following conditions be satisfied:*

(i) *In the neighborhood $\mathcal{O}_{u_0} \subset \mathcal{U}_M$ of point u_0 the following relation holds*

$$0 = u_0^{01} + (I - P_R)(G(u^{00} + u_0^{01} + u^1) + g(t)); \tag{9}$$

(ii) *Projector $P_R \in \mathcal{L}(\mathcal{U}_M^0)$, and operator $I + P_R G'_{u_0^0} : \mathcal{U}_M^{00} \to \mathcal{U}_M^{00}$ is a topological linear isomorphism ($\mathcal{U}_M^{00} = \mathcal{U}_M \cap \mathcal{U}^{00}$);*

(iii) *For the analytic semigroup $\{U_1^t : t \geq 0\}$ the following condition is valid*

$$\int_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1;\mathcal{U}_M^1)} dt < \infty \qquad \forall \tau \in \mathbb{R}_+. \tag{10}$$

Then there exists a unique solution of problem (3), (4), *which is the quasi-stationary semi-trajectory of equation* (4).

Proof. Consider the neighborhood $\mathcal{O}_{u_0}$ of point u_0. In this neighborhood the first equation (6) turns to

$$0 = u^{00} + P_R(G(u^{00} + u_0^{01} + u^1) + g(t)) \tag{11}$$

by condition (i). Further, from (i) by the implicit function theorem there exist neighborhoods $\mathcal{O}_{u_0^{00}} \subset \mathcal{U}_M^{00}$ ($\mathcal{U}_M^{00} = \mathcal{U}^{00} \cap \mathcal{U}_M$), $\mathcal{O}_{u_0^1} \subset \mathcal{U}_M^1$ ($\mathcal{U}_M^1 = \mathcal{U}^1 \cap \mathcal{U}_M$) of points $u_0^{00} = P_R(I - P)u_0,\ u_0^1$ respectively and the mapping $\delta : \mathcal{O}_{u_0^1} \to \mathcal{O}_{u_0^{00}}$ of class $\mathcal{C}^\infty$ such that the equation

$$u^{00} = \delta(u^1, t) \tag{12}$$

is equivalent to (11).

Now, by (12) the second equation in (6) in the neighborhood of $\mathcal{O}_{u_0^1}$ turns to

$$\dot{u}^1 = Su^1 + H(\delta(u^1) + u_0^{01} + u^1) + h(t), \tag{13}$$

where operator $H((I+\delta)(\cdot) + u_0^{01}) : \mathcal{O}_{u_0^1} \to \mathcal{U}^1$ belongs to class $\mathcal{C}^\infty$ by construction.

To prove the unique solvability of the problem $u^1(0) = u_0^1$ for equation (13) we use the Sobolevskiy–Tanabe method, described in [17, Chapter 9]. By (iii), smoothness of operator H and vector-function h all conditions of theorems 9.4, 9.6 and 9.7 in [17] holds. Therefore, if $u_0^1 \in \mathcal{U}_M^1$, then for some $T \in \mathbb{R}_+$ equation (13) has a unique solution $u^1 = u^1(t),\ t \in [0,T)$ such that $u^1(t) \to u_0^1$ for $t \to 0+$ in the topology of $\mathcal{U}_M^1$.

Thus, the solution of (3), (4) in this case has the form $u = u^1 + \delta(u^1) + u_0^{01}$, and this solution is a quasi-stationary semi-trajectory by construction.

□

Remark 6. For any quasi-stationary semi-trajectory of equation (4) relation (9) follows immediately from the first equation in (8).

Remark 7. Condition (10) for the conventional analytic semigroups with the estimate

$$\|U_1^t\|_{\mathcal{L}(\mathcal{U}^1;\mathcal{U}_M^1)} < t^{-1}\mathrm{const},$$

is not satisfied. Later we are going to use theorem 2 in such situation, therefore we need to make some explanations. Let $\mathcal{U}_\alpha^1 = [\mathcal{U}^1;\mathcal{U}_M^1]_\alpha,\quad \alpha \in [0,1]$ be some interpolation space constructed by the operator S. Supplement the condition "operator $F \in \mathcal{C}^\infty(\mathcal{U}_M^1;\mathcal{F})$, vector-function $f \in \mathcal{C}^\infty(\bar{\mathbb{R}}_+;\mathcal{F})$" in theorem 2 with the condition "operator $H \in \mathcal{C}^\infty(\mathcal{U}_M^1;\mathcal{U}_\alpha^1)$, $h \in \mathcal{C}^\infty(\mathbb{R}_+;\mathcal{U}_\alpha^1)$", and replace (10) with estimate

$$\int_0^\tau \|U_1^t\|_{\mathcal{L}(\mathcal{U}^1;\mathcal{U}_\alpha^1)} dt < \infty \qquad \forall \tau \in \mathbb{R}_+. \tag{14}$$

Then theorem 2 is true. See discussion of these problems in [17, Chapter 9].

Remark 8. Let the conditions of theorem 2 be satisfied (possibly taking into account the remark 7). Construct the plane $\mathcal{A} = \{u \in \mathcal{U}_M : (I-P_R)(I-P)u = u_0^{01}\}$ and the set $\mathcal{M} = \{u \in \mathcal{U}_M : P_R((I-P)u + G(u) + g(t)) = 0\}$.

By hypothesis, their intersection $\mathcal{A} \cap \mathcal{M} \neq \emptyset$, so it contains at least a point u_0. Moreover, there exists a $\mathcal{C}^\infty$-diffeomorphism $I+\delta$, mapping neighborhood $\mathcal{O}_{u_0^1}$ to some neighborhood $\mathcal{O}_{u_0} \subset \mathcal{A} \cap \mathcal{M}$. Consequently, not only point u_0 can be taken as the initial value, it may be an arbitrary point of neighborhood $\mathcal{O}_{u_0}$. This means that $\mathcal{O}_{u_0}$ is the part of *extended phase space* $\mathcal{B}^t$ of equation (4).

Now let $\mathcal{U}_k$ and $\mathcal{F}_k$ be Banach spaces, operators $A_k \in \mathcal{L}(\mathcal{U}_k, \mathcal{F}_k)$, and operators B_k : $\operatorname{dom} B_k \to \mathcal{F}$ be linear and closed with domains $\operatorname{dom} B_k$ dense in $\mathcal{U}_k$, $\quad k = 1,2$. Construct spaces $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\quad \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and operators $L = A_1 \otimes A_2$, $\quad M = B_1 \otimes B_2$. By construction operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, and operator $M :\quad \operatorname{dom} M \to \mathcal{F}$ is linear, closed and densely defined, in $\mathcal{U}$ $\operatorname{dom} M = \operatorname{dom} B_1 \times \operatorname{dom} B_2$.

Theorem 3. *[18] Let operators B_k be strongly (A_k, p_k)-sectorial, $k = 1,2$; and $p_1 \geq p_2$. Then operator M is strongly (L, p_1)-sectorial.*

2. The Concrete Interpretation

Consider problem (2) for system (1) given by

$$\begin{cases} (1-\lambda\nabla^2)\mathbf{v}_t = \nu\nabla^2\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{p} - g\gamma\theta + \mathbf{f}, \\ 0 = \nabla(\nabla\cdot\mathbf{v}), \\ \theta_t = æ\nabla^2\theta - \mathbf{v}\cdot\nabla\theta + \mathbf{v}\cdot\gamma. \end{cases} \tag{15}$$

Here $\mathbf{p} = \nabla p$, since in many hydrodynamic problems knowledge of the pressure gradient is preferable [19]. This change in the continuity equation was made for the first time by G.A. Sviridyuk in [20]. We are interested in a local unique solvability of problem (15), (2), equivalent to the original problem (1), (2). It's convenient to consider this problem in the frame of the semilinear Sobolev type equations theory, briefly described in section 2.

In order to reduce problem (15), (2) to (3), (4) we introduce, following [20], the spaces $\mathbf{H}^2_\sigma$, $\mathbf{H}^2_\pi$, $\mathbf{H}_\sigma$ and $\mathbf{H}_\pi$. Here $\mathbf{H}^2_\sigma$ and $\mathbf{H}_\sigma$ are subspaces of solenoidal functions in spaces $(W^2_2(\Omega))^n \cap (\overset{\circ}{W}{}^1_2(\Omega))^n$ and $(L_2(\Omega))^n$ respectively, and $\mathbf{H}^2_\pi$ and $\mathbf{H}_\pi$ are their orthogonal (in sense of $(L_2(\Omega))^n$) complements. By symbol Σ denote the orthogonal projection on $\mathbf{H}_\sigma$, where its restriction to space $(W^2_2(\Omega))^n \cap (\overset{\circ}{W}{}^1_2(\Omega))^n$ is denoted by the same symbol. Let $\Pi = I - \Sigma$.

By formula $A = \nabla^2 E_n : \mathbf{H}^2_\sigma \oplus \mathbf{H}^2_\pi \to \mathbf{H}_\sigma \oplus \mathbf{H}_\pi$, ($E_n$ is the identity matrix of order n,) determines the continuous linear operator with the discrete finite spectrum $\sigma(A) \subset \mathbb{R}$, condensed only at $-\infty$. Formula $B : \mathbf{v} \to \nabla(\nabla\cdot\mathbf{v})$ determines the continuous linear surjective operator $B : \mathbf{H}^2_\sigma \oplus \mathbf{H}^2_\pi \to \mathbf{H}_\pi$ with the kernel $\ker B = \mathbf{H}^2_\sigma$.

Using natural isomorphism of the direct sum and Cartesian product of Banach spaces, introduce spaces $\mathcal{U}_1 = \mathbf{H}^2_\sigma \times \mathbf{H}^2_\pi \times \mathbf{H}_p$ and $\mathcal{F}_1 = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \mathbf{H}_p$, where $\mathbf{H}_p = \mathbf{H}_\pi$. Construct the operators

$$A_1 = \begin{pmatrix} \Sigma(I-\lambda A) & \Sigma(I-\lambda A) & O \\ \Pi(I-\lambda A) & \Pi(I-\lambda A) & O \\ O & O & O \end{pmatrix}, \quad B_1 = \begin{pmatrix} \nu\Sigma A & \nu\Sigma A & O \\ \nu\Pi A & \nu\Pi A & -I \\ O & B & O \end{pmatrix}.$$

Remark 9. Denote by A_σ the restriction of ΣA on $\mathbf{H}^2_\sigma$. By the Solonnikov–Vorovich–Yudovich theorem the spectrum $\sigma(A_\sigma)$ is real, discrete, with finitely-multiplicities one and condenses only at $-\infty$.

Theorem 4. (i) *The operators* $A_1, B_1 \in \mathcal{L}(\mathcal{U}_1; \mathcal{F}_1)$. *If* $\lambda^{-1} \notin \sigma(A)$, *then operator* A_1 *is bi-splitting,* $\ker A_1 = \{0\} \times \{0\} \times \mathbf{H}_p$, $\operatorname{im} A_1 = \mathbf{H}_\sigma \times \mathbf{H}_\pi \times \{0\}$.

(ii) *If* $\lambda^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$, *then operator* B_1 *is an* $(A_1, 1)$*-bounded one.*

Remark 10. The proof of theorem 4 is given in [9]. For the first time the concept of relatively bounded operator was introduced in [21]. The case of relatively sectorial operator was considered in [7, 22, 23].

Set $\mathcal{U}_2 = \mathcal{F}_2 = L_2(\Omega)$ and by formula $B_2 = æ\nabla^2 : \operatorname{dom} B_2 \to \mathcal{F}_2$ determine the closed linear and densely defined operator B_2, $\operatorname{dom} B_2 = W^2_2(\Omega) \cap \overset{\circ}{W}{}^1_2(\Omega)$. If operator A_2 is equal to I, then by the sectoriality of operator B_2 [24, Chapter 1] the following theorem is valid.

Theorem 5. *The operator* B_2 *is a strongly* A_2*-sectorial one.*

Let $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. The vector u of space $\mathcal{U}$ has the form $u = \text{col}\,(u_\sigma, u_\pi, u_p, u_\theta)$, where $\text{col}\,(u_\sigma, u_\pi, u_p) \in \mathcal{U}_1$, and $u_\theta \in \mathcal{U}_2$. The vector $f \in \mathcal{F}$ has the same form. Operators L and M are defined by formulas $L = A_1 \otimes A_2$ and $M = B_1 \otimes B_2$. Operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$, and operator $M : \text{dom}\, M \to \mathcal{F}$ is linear, closed and densely defined, $\text{dom}\, M = \mathcal{U}_1 \times \text{dom}\, B_2$. From theorem 4 and [10] it follows that operator B_1 is strongly $(A_1, 1)$-sectorial. Therefore, by theorems 3 and 5 the following theorem is true.

Theorem 6. *Let $\lambda^{-1} \notin \sigma(A)$, then operator M is a strongly $(L, 1)$-sectorial one.*

We proceed to the construction of nonlinear operator F. In this case it is convenient to represent it in the form $F = F_1 \otimes F_2$, where $F_1 = F_1(u_\sigma, u_\pi, u_\theta) = \text{col}\,(-\Sigma(((\tilde{u}_\sigma + \tilde{u}_\pi)\cdot\nabla)(u_\sigma + u_\pi) - ((u_\sigma + u_\pi)\cdot\nabla)(\tilde{u}_\sigma + \tilde{u}_\pi) + g\gamma u_\theta + f), -\Pi(((\tilde{u}_\sigma + \tilde{u}_\pi)\cdot\nabla)(u_\sigma + u_\pi) - ((u_\sigma + u_\pi)\cdot\nabla)(\tilde{u}_\sigma + \tilde{u}_\pi) + g\gamma u_\theta + f), 0)$, and $F_2 = F_2(u_\sigma, u_\pi, u_\theta) = (u_\sigma + u_\pi)\cdot(\gamma - \nabla u_\theta)$.

Formally, we find the Frechet derivative F'_u of operator F at point u,

$$F'_u = \begin{pmatrix} \Sigma a(u_\sigma, u_\pi) & \Sigma a(u_\sigma, u_\pi) & O & -g\Sigma\gamma \\ \Pi a(u_\sigma, u_\pi) & \Pi a(u_\sigma, u_\pi) & O & -g\Pi\gamma \\ O & O & O & O \\ (\gamma - \nabla u_\theta)\cdot(*) & (\gamma - \nabla u_\theta)\cdot(*) & O & -(u_\sigma + u_\pi)\cdot(*) \end{pmatrix},$$

where $a(u_\sigma, u_\pi) = -((*)\cdot\nabla)(u_\sigma + u_\pi) - ((u_\sigma + u_\pi)\cdot\nabla)(*)$, and the character * should be changed the corresponding coordinate of vector v in case of finding a vector $F'_u v$.

Further, the space $\mathcal{U}_M = \mathcal{U}_1 \times \text{dom}\, B_2$ (by the continuity operator B_1). Using a standard technique (see, e.g., [13, 14]), it is easy to show that for arbitrary $u \in \mathcal{U}_M$ operator $F'_u \in \mathcal{L}(\mathcal{U}_M;\mathcal{F})$. Similarly we establish that the second Frechet derivative F''_u of operator F is a continuous bilinear operator from $\mathcal{U}_M \times \mathcal{U}_M$ to $\mathcal{F}$, and $F'''_u \equiv O$. So the following theorem is valid.

Theorem 7. *The operator $F \in \mathcal{C}^\infty(\mathcal{U}_M;\mathcal{F})$.*

The vector-function $f = f_1 \otimes f_2$, where $f_1 = \text{col}(\Sigma\mathbf{f}^1, \Pi\mathbf{f}^1, 0)$, $f_2 = 0$. We assume that $f \in \mathcal{C}^\infty(\bar{\mathbb{R}}_+;\mathcal{F})$.

Thus, the reduction of problem (15), (2) to (3), (4) is finished. Further we identify problems (15), (2) and (3), (4).

Now let's check the conditions of theorems 1 and 2. By theorem 6 and the corresponding results of [10] there exists the analytic semigroup $\{U^t : t \in \mathbb{R}_+\}$ of the resolving operators for equation (4) which is in this case naturally presented in the form $U^t = V^t \otimes W^t$, where $V^t(W^t)$ is the restriction of operator U^t on $\mathcal{U}_1$ ($\mathcal{U}_2$). Since the operator B_2 is sectorial, then $W^t = \exp(tB_2)$, which leads to $\mathcal{W}^0 = \{0\}$ and $\mathcal{W}^1 = \mathcal{U}_2$.

Consider the semigroup $\{V^t : t \in \mathbb{R}_+\}$. By theorems 4, 6 and monograph [10] this semigroup can be extended to the group $\{V^t : t \in \mathbb{R}\}$. Its kernel $\mathcal{V}^0 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$ where $\mathcal{U}_1^{00} = \{0\} \times \{0\} \times \mathbf{H}_p(= \ker A_1$ (by theorem 4), and $\mathcal{U}_1^{01} = \Sigma A_\lambda^{-1} A_{\lambda\pi}^{-1}[\mathbf{H}_\pi^2] \times \mathbf{H}_\pi^2 \times \{0\}$. Here $A_\lambda = I - \lambda A$, $A_{\lambda\pi}$ is the restriction of operator ΠA_λ^{-1} to $\mathbf{H}_\pi$. It is known that if $\lambda^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$, then operator $A_{\lambda\pi} : \mathbf{H}_\pi \to \mathbf{H}_\pi^2$ is topological linear isomorphism (see, for example, [9]). Let $\mathcal{U}_1^1$ be the image of $\mathcal{V}^1$. Then the space $\mathcal{U}_1$ can be decomposed into the direct sum of subspaces: $\mathcal{U}_1 = \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01} \oplus \mathcal{U}_1^1$.

Construct the operator $R = B_{10}^{-1} A_{10} \in \mathcal{L}(\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$, where $A_{10}\,(B_{10})$ is the restriction of operator $A_1\,(B_1)$ on $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$. (The operator B_{10}^{-1} exists the theorem 6 and the corresponding results of [10]).

Obviously, $\ker R = \mathcal{U}_1^{00}$, and it is shown [20] that $\operatorname{im} R = \mathcal{U}_1^{00}$. Hence, the operator R is bi-splitting. Let P_R be a projector in the space $\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01}$ on $\mathcal{U}_1^{00}$ along $\mathcal{U}_1^{01}$. By construction of space $\mathcal{U}_M$ projector $P_R \in \mathcal{L}(\mathcal{U}_M^0)$, where $\mathcal{U}_M^0 = \mathcal{U}_M \cap (\mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})(\equiv \mathcal{U}_1^{00} \oplus \mathcal{U}_1^{01})$.

Lemma 1. *Let $\lambda^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then the operator R is bi-splitting, and $P_R \in \mathcal{L}(\mathcal{U}_M^0)$.*

Introduce the projectors

$$P_0 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & O & \Pi \end{pmatrix}, \qquad P_1 = \begin{pmatrix} O & P_1^{12} & O \\ O & \Pi & O \\ O & O & O \end{pmatrix},$$

where $P_1^{12} = \Sigma A_\lambda^{-1} A_{\lambda\pi}^{-1}$. From [20] since the kernel $\mathcal{W}^0 = \{0\}$, projector $I - P = (P_0 + P_1) \otimes O$. Applying projector $I - P$ to the equation (4) in this transcription we obtain

$$\begin{gathered} \Pi(\nu A(u_\sigma + u_\pi) - ((u_\sigma + u_\pi) \cdot \nabla)(u_\sigma + u_\pi) - u_p - g\gamma u_\theta + f(t)) = 0, \\ B u_\pi = 0. \end{gathered} \tag{16}$$

Hence, by theorem 1 and properties of operator B we obtain the necessary condition for quasi-stationarity of the semi-trajectory $u_\pi \equiv 0$. In other words, all solutions of the problem (if they exist) need to lie in the plane $\mathcal{B} = \{u \in \mathcal{U}_M : u_\pi = 0\}$.

But since $\Pi u_p = u_p$, from the first equation in (16) we obtain relation (8) in a transcription

$$u_p = \Pi(\nu A u_\sigma - (u_\sigma \cdot \nabla) u_\sigma - g\gamma u_\theta + f(t)). \tag{17}$$

Obviously, $P_0 \equiv P_R$, so the second equation in (16) is relation (9) with respect to our situation. So, we have

Lemma 2. *Under the conditions of lemma 1, any solution of* (3), (4) *belong to the set*

$$\mathcal{A}^t = \{u \in \mathcal{U}_M : u_\pi = 0, \quad u_p = \Pi(\nu A u_\sigma - (u_\sigma \cdot \nabla) u_\sigma - g\gamma u_\theta) + f_\pi(t)\}.$$

Remark 11. Relation (17) immediately implies the condition (iii) of theorem 2 for every point $u_0^0 \in \mathcal{U}_M^{00} (\equiv \mathcal{U}_1^{00} \times \{0\})$. Therefore, by remark 8 the set $\mathcal{A}^t$ (a simple Banach $\mathcal{C}^\infty$-manifold diffeomorphic to the subspace $\mathcal{U}_1^1 \times \mathcal{U}_2$) is the candidate for extended phase space of problem (15), (2).

We proceed to verify conditions (10) and (14). Construct the space $\mathcal{U}_\alpha = \mathcal{U}_1 \times \overset{\circ}{W}{}_2^1(\Omega)$. This space is obviously the interpolation space for the pair $[\mathcal{U}, \mathcal{U}_M]_\alpha$, with $\alpha = 1/2$. As noted above, the semigroup $\{U^t : t \in \mathbb{R}_+\}$ extends to a group $\{V_1^t : t \in \mathbb{R}\}$ on $\mathcal{U}_1^1$, where V_1^t is the restriction of the operator V^t on $\mathcal{U}_1^1$. Since $\mathcal{U}_M^1 = \mathcal{U}_M \cap \mathcal{U}_1^1$ (by construction) and operator B_1 is continuous (theorem 4), then by the uniform boundedness of semigroup $\{U^t : t \in \mathbb{R}_+\}$ we have

$$\int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1; \mathcal{U}_M^1)} dt \leq \mathrm{const} \|B_1\|_{\mathcal{L}(\mathcal{U}_1; \mathcal{F}_1)} \int_0^\tau \|V_1^t\|_{\mathcal{L}(\mathcal{U}_1^1)} dt < \infty \qquad \forall \tau \in \mathbb{R}_+. \tag{18}$$

Further, by Sobolev inequality [17, Chapter 9] semigroup $\{W^t : t \in \bar{\mathbb{R}}_+\}$ satisfies

$$\int_0^\tau \|W^t\|_{\mathcal{L}(\operatorname{dom} B_2;\, \overset{\circ}{W}{}_2^1(\Omega))} dt < \infty. \tag{19}$$

Let $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$, where $\mathcal{U}^1 = \mathcal{U}_1^1 \times \mathcal{U}_2$. Then from (18) and (19) imply the following

Lemma 3. *Under the conditions of lemma 1, relation* (10) *is fulfilled.*

Finally, for checking the requirement (14), we should find the operator H and the vector-function h. For this purpose we construct the projector $Q : \mathcal{F} \to \mathcal{F}^1$. According to [20] $Q = (I - Q_0 - Q_1) \otimes I$, where

$$Q_0 = \begin{pmatrix} O & O & O \\ O & \Pi & Q_0^{23} \\ O & O & O \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} O & O & Q_1^{13} \\ O & O & Q_1^{23} \\ O & O & \Pi \end{pmatrix},$$

$Q_1^{13} = \Sigma A A_\lambda^{-1} A_{\lambda\pi}^{-1} B_\pi^{-1}$, $Q_1^{23} = \Pi A A_\lambda^{-1} A_{\lambda\pi}^{-1} B_\pi^{-1}$, $Q_0^{23} = -Q_1^{23}$, and operator B_π is a restriction of operator B on $\mathbf{H}_\pi^2$ (by Banach theorem about the inverse operator the operator $B_\pi : \mathbf{H}_\pi^2 \to \mathbf{H}_\pi$ is a toplinear isomorphism). So, the operator $H = H_1 \otimes H_2$, where $H_1 = A_{11}^{-1}(I - Q_0 - Q_1)F_1$, and $H_2 = F_2$ (A_{11} is the restriction of A on $\mathcal{U}_1^1$).

The inclusion $H \in \mathcal{C}^\infty(\mathcal{U}_M^1; \mathcal{U}_\alpha^1)$, where $\mathcal{U}_\alpha^1 = \mathcal{U}_\alpha \cap \mathcal{U}^1$ is shown in the same way inclusion $F \in \mathcal{C}^\infty(\mathcal{U}_M; \mathcal{F})$. Vector-function $h(t)$ is defined as $h_1(t) \otimes h_2(t)$, where $h_1 = A_{11}^{-1}(I - Q_0 - Q_1)f_1$, and $h_2 = 0$.

The vector-function $f = f_1 \otimes f_2$ has the infinite smoothness by construction. So $h \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{U}_\alpha^1)$.

Thus, all conditions of theorem 2 are satisfied. Therefore we have

Theorem 8. *Let $\lambda^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then for any u_0 such that $(u_0, 0) \in \mathcal{A}^0$ and some $T \in \mathbb{R}_+$ there exists a unique solution $u = (u_\sigma, 0, u_p, u_\theta)$ to problem* (1), (2), *which is a quasi-stationary semi-trajectory, and $(u(t), t) \in \mathcal{A}^t$, for all $t \in (0, T)$.*

Other non-stationary models of the incompressible viscoelastic fluids are considered, in [9, 25–30]. Linearized models of different orders were studied in [31–36].

The non-autonomous case is described in detail in [37]. The Taylor problem for the generalized model of the non-zero order is studied in [38]. Different models of non-zero order in the autonomous case are studied in [39]. Investigation of magnetohydrodynamic models using the semigroup approach was initiated in [40, 41].

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ОБ ОДНОМ КЛАССЕ УРАВНЕНИЙ СОБОЛЕВСКОГО ТИПА

Т.Г. Сукачева, А.О. Кондюков

Статья содержит обзор работ Т.Г. Сукачевой и ее учеников в области исследования моделей несжимаемых вязкоупругих жидкостей Кельвина – Фойгта в рамках теории полулинейных уравнений соболевского типа. Основное внимание уделяется нестационарному случаю ввиду его большей общности. Идея исследования демонстрируется на примере нестационарной задачи термоконвекции для модели Осколкова нулевого порядка. Вначале изучается абстрактная задача Коши для полулинейного неавтономного уравнения соболевского типа, а затем соответствующая начально-краевая задача рассматривается как конкретная интерпретация этой задачи. Доказана теорема существования единственного решения указанной задачи, являющегося квазистационарной полутраекторией, и получено описание ее расширенного фазового пространства. Подобным образом могут быть исследованы и другие задачи гидродинамики, например, линеаризованные модели Осколкова, задача Тейлора, а также некоторые модели, описывающие движение несжимаемых вязкоупругих жидкостей Кельвина – Фойгта в магнитном поле Земли.

Ключевые слова: уравнения соболевского типа; несжимаемая вязкоупругая жидкости; относительно p-секториальные операторы; расширенное фазовое пространство.

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