

INTRODUCING A POWER OF THE OPERATOR IN DIRECT SPECTRAL PROBLEMS

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The resolvent method, proposed by Sadovnichiy and Dubrovsky in the 1990s, is successfully applied in the direct spectral problem to calculate the asymptotics of eigenvalues of the perturbed operator, find formulas for the regularized trace, and recover perturbation. But the application of this method faces difficulties when the resolvent of the unperturbed operator is non-nuclear. Therefore, a number of physical problems could only be considered on the interval. This article describes a justification of the transition to the power of an operator in order to expand the area of possible applications of the resolvent method. Considering the problem of calculating the regularized trace of the Laplace operator on a parallelepiped of arbitrary dimension, we show that for every fixed dimension it is possible to choose the required power of the operator and to calculate the regularized traces. These studies are relevant due to the need to study important applied problems, particularly in hydrodynamics, electronics, elasticity theory, quantum mechanics, and other fields.

Keywords: regularized trace; Laplace operator; power of operator.

Introduction

Even though the spectral theory of operators began in the nineteenth century, it is still actively developing. It includes separate directions like the problem of computing regularized trace, inverse problems, the calculation of spectral characteristics. This work, representing the school of Sadovnichiy and Dubrovsky, is devoted to the calculation of the regularized trace of the Laplace operator on a parallelepiped of arbitrary dimension. The problem of finding the regularized trace, besides its mathematical significance, has physical meaning. For example, the formula for the regularized trace of the Sturm–Liouville equation describes energy conservation in the dynamical system described by the Korteweg–de Vries equation [1]. With the aid of regularized traces calculated by Liphshic [2], a modification of the free energy of a crystal with implemented admixtures was found.

The theory of regularized traces originated with the work of Gelfand and Levitan (1953) [3], who found the asymptotics of the eigenvalues of the Sturm–Liouville operator. In the 1960s Lidskii and Sadovnichiy almost completed this theory for ordinary differential operators. They succeeded in calculating the regularized traces of boundary value problems for ordinary differential equations of arbitrary order with a complex occurrence of the parameter. Partial differential operators are much more difficult to study. Kostyuchenko and Gasimov [4], as well as other authors, obtained various results in this direction. The challenge is that for partial differential equations the resolvent has special structure, and the exact asymptotics of the eigenvalues is unknown. Sadovnichiy and Dubrovsky proposed

the so-called resolvent method, which enabled them to avoid these difficulties using perturbative corrections. Subsequently, basing on this method, the followers of Dubrovsky obtained a regularized trace formula for the Laplace operator [5] or a Chebyshev-type operator [6] and investigated the existence and uniqueness [7] of the perturbing operator and the stability of the solution [8]. They proposed algorithms for reconstructing the potential for models with the Laplace operator [9]. The approach of these studies rests on the principle of lowering the power of the operator, as the Laplace operator itself is widely applied in physics rather than its powers. This paper presents a fundamentally different approach: we regard the power of an operator as an opportunity to explore physical problems not only on the interval, but also in other dimensions.

1. The Regularized Trace

Take the N -dimensional parallelepiped

$$\Pi = \{x = (x_1, x_2, \dots, x_N) : 0 \leq x_j \leq a_j, j = 1, \dots, N\},$$

where $a_j > 0$. Denote by \mathcal{U} a separable Hilbert space with the inner product

$$(f, h) = \int_{\Pi} f(x)h(x)dx$$

and the norm $\|h\| = \sqrt{(h, h)}$. In the space $\mathcal{L}(\mathcal{U})$ consider the discrete selfadjoint operator T of the Dirichlet boundary value problem

$$-Tv = \lambda v, \quad v|_{\partial\Pi} = 0.$$

By a discrete operator we understand an operator whose spectrum is discrete. Assume that the eigenvalues $\lambda_m \in \mathbf{R}$ of T , for $m = (m_1, m_2, \dots, m_N)$, in the ascending order, have the asymptotics $\lambda_k \sim Ck^{\frac{2\beta}{N}}$ with $\beta \geq 1$ and $C = const$. For example, consider the Dirichlet boundary value problem for the Laplace operator on the rectangle. Obviously, the eigenvalues in this case satisfy the asymptotics. [10]

Next, given a bounded operator P defined everywhere on \mathcal{U} , we can show that $T + P$ is a discrete operator. Denote by μ_k its eigenvalues in the ascending order of their real parts.

Lemma 1. *Take a discrete selfadjoint operator $T \in \mathcal{L}(\mathcal{U})$. If the perturbing operator P satisfies $\|P\| < r/2$, where $0 < r \leq r_0 = \inf_k r_k$ with $r_k = \frac{1}{2} \min\{\lambda_{k+1} - \lambda_k; \lambda_k - \lambda_{k-1}\}$ for $k > 1$ and $r_1 = \frac{1}{2}(\lambda_2 - \lambda_1)$, then $T + P$ is discrete and the following claims hold:*

- (i) if $R_0(\lambda) \in S/q$ then $R(\lambda) \in S/q$ for $1 \leq q < \infty$;
- (ii) if $\lambda_k \in C \setminus \Omega_{r_k}$ then $\mu_k^s \in C \setminus \Omega_{r_k}$ for $s = \overline{1, \nu_k}$, and ν_k is the multiplicity of the eigenvalue λ_k .

Therefore, the multiplicities of the eigenvalues of T and $T + P$ lying in the circle γ_{r_k} are equal.

Let us put the problem of finding the numbers A_k so that the series

$$\sum_k (\mu_k - \lambda_k - A_k)$$

converges and determining the sum of the series. This series is called the *regularized trace* of the operator $T + P$.

As we can see from the asymptotics, for all $N \geq 2$ the series

$$\sum_{t=0}^{\infty} \frac{1}{\lambda_t}$$

diverges. Therefore, we cannot apply the resolvent method when $N > 2$. To resolve this difficulty and treat arbitrary dimension, we introduce a new boundary value problem with the power T^β of T :

$$(-T^\beta + P)v = \lambda v, \quad v|_{\partial\Pi} = 0.$$

The eigenfunctions of these problems coincide, but the eigenvalues of T^β are equal to λ_k^β . The asymptotics of the eigenvalues of the new operator shows that it is possible to consider the problem in arbitrary dimension when a suitable k is found.

Assume that the suitable k is found. Following[9], we obtain

Theorem 1. *If $\beta > N/2$ and $\|P\| < r/2$, where $0 < r \leq r_0$, then for all $k \in \mathbb{N}$ we have*

$$\sum_{s=1}^{\nu_k} \mu_k^s = \nu_k \lambda_k^\beta - \sum_{s=1}^{\nu_k} (Pv_k^s, v_k^s) + \alpha_k(p),$$

where

$$\alpha_k(p) = \frac{1}{2\pi i} \int_{\gamma_{r_k}} \lambda \operatorname{Tr} \left[R(\lambda) \left(PR_0(\lambda) \right)^2 \right] d\lambda.$$

Evaluating perturbative corrections, we see that they vanish for all $n > 2$. Thus, we obtain

Theorem 2. *If $\beta > N/2$ and $\|P\| < r/2$, where $0 < r \leq r_0$, then for all $k \in \mathbb{N}$ we have the regularized trace formula*

$$\sum_{s=1}^{\nu_k} \mu_k^s = \nu_k \lambda_k^\beta - \sum_{s=1}^{\nu_k} (Pv_k^s, v_k^s).$$

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ВВЕДЕНИЕ СТЕПЕНИ ОПЕРАТОРА ПРИ РЕШЕНИИ ПРЯМЫХ СПЕКТРАЛЬНЫХ ЗАДАЧ

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Резольвентный метод, предложенный еще в 90-х гг В.А. Садовничим и В.В. Дубровским, с успехом применим как в прямых спектральных задачах при вычислении асимптотики собственных чисел возмущенного оператора или формул регуляризованных следов, так и в обратных – при восстановлении потенциала. Однако, применение этого метода вызывает затруднения в тех случаях, когда резольвента невозмущенного оператора оказывается неядерной. Поэтому ряд физических задач, как известно, приходится рассматривать только на интервале. В данной работе приведено обоснование перехода к степени оператора для расширения области применения резольвентного метода. Рассмотрен вопрос о вычислении регуляризованного следа оператора Лапласа на параллелепипеде произвольной размерности. Показано, что для любой фиксированной размерности возможно подобрать нужную степень оператора и вычислить регуляризованный след. Актуальность этих исследований обусловлена необходимостью изучения важных прикладных задач, в частности, в области гидродинамики, радиоэлектроники, теории упругости, квантовой механики и других.

Ключевые слова: регуляризованный след; оператор Лапласа.

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