A MULTIPOINT INITIAL-FINAL VALUE PROBLEM FOR A LINEAR MODEL OF PLANE-PARALLEL THERMAL CONVECTION IN VISCOELASTIC INCOMPRESSIBLE FLUID

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The linear model of plane-parallel thermal convection in a viscoelastic incompressible Kelvin–Voigt material amounts to a hybrid of the Oskolkov equations and the heat equations in the Oberbeck–Boussinesq approximation on a two-dimensional region with Bénard's conditions. We study the solvability of this model with the so-called multipoint initial-final conditions. We use these conditions to reconstruct the parameters of the processes in question from the results of multiple observations at various points and times. This enables us, for instance, to predict emergency situations, including the violation of continuity of thermal convection processes as a result of breaching technology, and so forth.

For thermal convection models, the solvability of Cauchy problems and initial-final value problems has been studied previously. In addition, the stability of solutions to the Cauchy problem has been discussed. We study a multipoint initial-final value problem for this model for the first time. In addition, in this article we prove a generalized decomposition theorem in the case of a relatively sectorial operator. The main result is a theorem on the unique solvability of the multipoint initial-final value problem for the linear model of plane-parallel thermal convection in a viscoelastic incompressible fluid.

Keywords: multipoint initial-final value problem; Sobolev-type equation; generalized splitting theorem; linear model of plane-parallel thermal convection in viscoelastic incompressible fluid.

Many phenomena and processes in economics, physics, and technology, like, for instance, plane-parallel thermal convection in viscoelastic incompressible fluid, are modelled by linear

$$L\dot{u} = Mu + f \tag{1}$$

and nonlinear

$$L\dot{u} = Mu + N(u) + f \tag{2}$$

Sobolev-type equations [33]. The interest in Sobolev-type equations, which nowadays form a large subfield of nonclassical equations of mathematical physics [30], has been increasing recently; see the wonderful historical survey in [31].

The goal of our study is the solvability of (1) with the so-called *multipoint initial-final conditions* (see [5] for instance and reference therein)

$$P_j(u(\tau_j) - u_j) = 0, \quad u_j \in \mathfrak{U}, \quad j = \overline{0,n},$$
$$-\infty \le a < \tau_0 < \tau_1 < \ldots < \tau_j < \tau_{j+1} < \ldots < b \le +\infty, \tag{3}$$

where P_j are *relative spectral projectors* (we discuss them in Section 4), while u_j are arbitrary vectors in a Banach space $\mathfrak{U}$. These conditions are used to reconstruct the parameters of the

processes in question from the results of multiple observations at various points and times. This enables us to, for instance, predict emergency situations, including the violation of continuity of thermal convection processes as a result of breaching technology, and so forth.

We should note that problem (1), (3) in the case $n = 1$ (the initial-final value problem) has been studied quite actively in various aspects. In particular, there are results concerning the optimal control of the solution to these problems [9], including Sobolev-type equations of high order [6].

The history of problem (1), (3) in the case $n = 1$ starts on the one hand in [12], where it is called Verigin's problem, and on the other hand, independently, in [32], where it is called the conjugation problem. However, in both cases instead of relatively spectral projectors P_0 and P_1 we consider spectral projectors of the operator L on assuming it to be selfadjoint. The first results in this direction are presented in [19], which treats a particular case of problem (1), (3) with, moreover, more rigid conditions on the L-spectrum of M than here. Problem (1), (3) is considered in [2] with the same conditions on the L-spectrum of M as in [19]; however, in this case the possibility of greater freedom in relatively spectral conditions is mentioned. We should note that there were attempts to study [18] the solvability of a particular case of problem (2) for nonlinear Sobolev-type equations (3) in the case $n = 1$, but as yet these studies have not been pushed further. In addition, if in (3) we put $n = 0$ then problem (3) reduces to the Showalter–Sidorov problem [20], which has already played an important role in a number of models with applications to economics [7] and technology [29].

Our approach rests on the theory of relatively p-sectorial operators and degenerate analytic resolving semigroups of operators. Sviridyuk [14] pioneered the concept of a relatively sectorial operator. He showed that relative sectoriality of an operator naturally generalizes the concept of sectoriality [28]. However, it soon turned out that relative sectoriality generalizes the concept of relative σ-boundedness of an operator only in the case that the L-resolvent of M has a removable singularity at infinity. In order to fill this embarrassing gap, Bokareva introduced [16] the concept of relative p-sectoriality of an operator, generalizing the concept of relative σ-boundedness in the case that the L-resolvent of M has a pole at infinity. Then, relatively strongly p-sectorial operators on the right (on the left) [16] and relatively strongly p-sectorial operators [24], [25] were introduced. Subsequently, relatively p-sectorial operators were studied in various situations. Namely, Dudko studied [1] the case that both operators are closed and the spaces $\mathfrak{U}$ and $\mathfrak{F}$ coincide; Efremov studied [17] optimal control problems for Sobolev-type equations with relatively p-sectorial operators; Keller found [21] sufficient, and in some cases necessary, conditions for the existence of bounded solutions to these equations; Kuznetsov [22] began to search for relatively p-sectorial operators among elliptic operators; Yakupov used [26] relatively p-sectorial operators to study the phase spaces of certain problems in the hydrodynamics of viscoelastic fluid.

Consider now a precursor of (2): a hybrid of Oskolkov's system [11] and the heat equation in the Oberbeck–Boussinesq approximation [8],

$$\begin{cases} (\lambda - \nabla^2)v_t = \nu\nabla^2 v - (v \cdot \nabla)v - \nabla p + g\gamma S, \ \nabla \cdot v = 0, \\ \qquad\qquad S_t = \delta\nabla S - v \cdot \nabla S + \gamma \cdot v, \end{cases} \tag{4}$$

modeling thermal convection in a viscoelastic incompressible Kelvin–Voigt material [23]. Here $v = (v_1, v_2, \cdots, v_n)$ with $v_i = v_i(x,t)$ and $n = 2$ or 3 is the vector function representing fluid velocity; the scalar functions $S = S(x,t)$ and $p = p(x,t)$ represent the temperature and pressure of the fluid; the parameters $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R}_+$, and $\delta \in \mathbb{R}_+$ characterize the elasticity, viscosity, and thermal conductivity of the fluid; $g \in \mathbb{R}_+$ is the free fall acceleration; finally, $\gamma = (0, \ldots, 0, 1) \in \mathbb{R}^n$ and $x = (x_1, x_2, \cdots, x_n)$. When one of the horizontal components of the velocity vanishes, (4)

becomes

$$
\begin{cases}
(\lambda - \Delta)\Delta\dfrac{\partial\psi}{\partial t} = \nu\Delta^2\psi - \dfrac{\partial(\psi, \Delta\psi)}{\partial(x, y)} + \alpha\dfrac{\partial\theta}{\partial x}, \\[3mm]
\dfrac{\partial\theta}{\partial t} = \delta\Delta\theta - \dfrac{\partial(\psi, \theta)}{\partial(x, y)} + \beta\dfrac{\partial\psi}{\partial x},
\end{cases}
\tag{5}
$$

which models plane-parallel thermal convection in a layer of viscoelastic incompressible Kelvin–Voigt material.

For (4) Sviridyuk considered the first initial-boundary value problem [13] and showed that it is solvable for arbitrary values of λ. Then jointly with Yakupov [26] he described the morphology of the phase space of the Cauchy–Bénard problem for (5). Sukacheva and Matveeva studied [27] the non-autonomous case of this problem. Subsequently they considered a generalized model of thermal convection [10], established the local solvability of the Cauchy problem for it, and found the solution numerically using the modified Galerkin method. We should also mention the studies [4] of the stability of solutions to the Cauchy–Bénard problem for (5) in a neighborhood of the origin. The existence of stable and unstable invariant manifolds in the problem was established basing on the Hadamard–Perron theorem. Note also that [3] showed the unique solvability of the initial-final value problem for the linearized model of thermal convection (4).

This article is devoted to a qualitative study of the multipoint initial-final value problem

$$
(\lambda - \Delta)\Delta\psi_t = -\nu\Delta^2\psi - \alpha\theta_x + \xi, \quad \theta_t = \delta\Delta\theta - \beta\psi_x + \zeta
\tag{6}
$$

for the linear mathematical model of plane-parallel thermal convection in viscoelastic incompressible fluid in the region $\Omega = (0, a) \times (0, b) \in \mathbb{R}^2$ with Bénard's boundary conditions

$$
\psi(x, 0, t) = \Delta\psi(x, 0, t) = \psi(x, b, t) = \Delta\psi(x, b, t) = 0,
\tag{7}
$$

$$
\theta(x, 0, t) = \theta(x, b, t) = 0,
\tag{8}
$$

the functions ψ and θ are periodic in x with period a. $\qquad$ (9)

In the first three sections we collect auxiliary facts of Sviridyuk's theory [15] of relatively p-sectorial operators and degenerate resolving semigroups of operators, adapted to our situation. In Section 1 we introduce the concept of a relatively p-sectorial operator. In Section 2 we consider degenerate resolving semigroups of operators and the construction of units of semigroups of operators. In Section 3 we consider conditions for the existence of the inverse operator. In Section 4 we prove a generalized splitting theorem for the spaces and actions of operators. There we construct relatively spectral projectors, which in this case are units of the semigroups of operators, on assuming relative p-sectoriality. In Section 5 we study the multipoint initial-final value problem for Sobolev-type equations with a relatively p-sectorial operator M. The main result of this section is a theorem on the unique solvability of problem (1), (3). In Section 6 we apply these abstract results to the linear model of plane-parallel thermal convection in viscoelastic incompressible fluid. There we reduce the stated problem to the abstract equation (1). We verify the $(L, 0)$-sectoriality of M. The main result of this section is the theorem on the unique solvability of the multipoint initial-final value problem (3). We should note that the author already discussed [34] a generalized decomposition theorem in the case of strongly (L, p)-radial operator. We make all arguments in real Banach spaces, but, while addressing spectral questions, introduce their natural complexifications. All contours are oriented counterclockwise and bound the region lying to the left as they are traversed.

1. Relatively p-sectorial Operators

On assuming that $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, consider a continuous linear operator $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and a closed linear operator $M \in \mathcal{Cl}(\mathfrak{U}; \mathfrak{F})$ whose domain is dense. Introduce the L-resolving set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and the L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of M. Provided that $\rho^L(M) \neq \emptyset$, we can introduce the *right* and *left*

$$R_{(\mu,p)}^L(M) = \prod_{k=0}^{p} R_{\mu_k}^L(M) \quad \text{and} \quad L_{(\mu,p)}^L(M) = \prod_{k=1}^{p} L_{\mu_k}^L(M)$$

(L, p)-*resolutions* of M. Here $R_\mu^L(M) = (\mu L - M)^{-1}L$ and $L_\mu^L(M) = L(\mu L - M)^{-1}$, while $\mu_k \in \rho^L(M)$ for $k = 0, \ldots, p$.

Definition 1. [15] An operator M is called p-*sectorial* relatively to an operator L with $p \in \{0\} \cup \mathbb{N}$ (or briefly, (L, p)-*sectorial*) whenever there exist constants $K \in \mathbb{R}_+$, $a \in \mathbb{R}$, and $\Theta \in (\pi/2, \pi)$ such that

$$S_{a,\Theta}^L(M) = \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \Theta, \ \mu \neq a\} \subset \rho^L(M); \tag{10}$$

furthermore,

$$\max\left\{\left\|R_{(\mu,p)}^L(M)\right\|_{\mathcal{L}(\mathfrak{U})}, \left\|L_{(\mu,p)}^L(M)\right\|_{\mathcal{L}(\mathfrak{F})}\right\} \leq \frac{K}{\prod\limits_{k=0}^{p} |\mu_k - a|} \tag{11}$$

for arbitrary $\mu_k \in S_{a,\Theta}^L(M)$ for $k = 0, \ldots, p$.

Remark 1. If M is an (L, p)-sectorial operator and $b \geq a$ then the operator $\tilde{M} = M - bL$ is also (L, p)-sectorial. Furthermore, we can choose the constant a in 1 to be 0. Assume henceforth that $S_{0,\Theta}^L(M) = S_\Theta^L(M)$.

Remark 2. If the operator $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ has continuous inverse then the sectoriality of the operator $L^{-1}M \in \mathcal{Cl}(\mathfrak{U})$ implies the (L, p)-sectoriality of $M \in \mathcal{Cl}(\mathfrak{U}; \mathfrak{F})$, and the $(L, 0)$-sectoriality of M implies the sectoriality of $L^{-1}M$ (or equivalently, of ML^{-1}).

Lemma 1. *If M is an (L, p)-sectorial operator then there exist $R > 0$ and $C > 0$ such that $\|(\mu L - M)^{-1}\|_{\mathcal{L}(\mathfrak{F}; \mathfrak{U})} \leq C|\mu|^p$ for all $\mu \in S_\Theta^L(M) \setminus \{\mu \in \mathbb{C} : |\mu| \leq R\}$.*

Remark 3. (i) If M is an (L, p)-bounded operator and ∞ is an order 0 pole of the L-resolvent of M then M is an $(L, 0)$-sectorial operator.

(ii) If M is an (L, p)-bounded operator and ∞ is a pole of order at most $p \in \mathbb{N}$ of the L-resolvent of M then M is an (L, p)-sectorial operator.

Lemma 2. *The following claims hold for every (L, p)-sectorial operator M:*

(i) the length of every chain of generalized M-eigenvectors of L is bounded by p;

(ii) the set $\ker R_{(\mu,p)}^L(M)$ coincides with the M-root space of L;

(iii) $\ker R_{(\mu,p)}^L(M) \cap \operatorname{im}R_{(\mu,p)}^L(M) = \{0\}$ and $\ker L_{(\mu,p)}^L(M) \cap \operatorname{im}L_{(\mu,p)}^L(M) = \{0\}$;

(iv) the operator $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ exists.

On assuming that M is an (L, p)-sectorial operator, recall the notation $H = M_0^{-1}L_0$ and $G = L_0 M_0^{-1}$. Denote by $\mathfrak{U}^1$ the closure of the linear subspace $\operatorname{im}R_{(\mu,p)}^L(M)$. Denote by $\tilde{\mathfrak{U}}$ the closure of the linear subspace $\mathfrak{U}^0 \dotplus \operatorname{im}R_{(\mu,p)}^L(M)$ in the norm of $\mathfrak{U}$. Denote by $\mathfrak{F}^1$ the closure of the linear subspace $\operatorname{im}L_{(\mu,p)}^L(M)$, and by $\tilde{\mathfrak{F}}$ the closure of the linear subspace $\mathfrak{F}^0 \dotplus \operatorname{im}L_{(\mu,p)}^L(M)$ in the norm of $\mathfrak{F}$.

Lemma 3. *The following claims hold for every (L,p)-sectorial operator M:*

(i) *the operators H and G are nilpotent of degree at most p;*

(ii) $\lim\limits_{\mu\to+\infty}(\mu R_\mu^L(M))^{p+1}u = u$ *for every* $u \in \mathfrak{U}^1$ *and* $\lim\limits_{\mu\to+\infty}(\mu L_\mu^L(M))^{p+1}f = f$ *for every* $f \in \mathfrak{F}^1$;

(iii) $\tilde{\mathfrak{U}} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ *and* $\tilde{\mathfrak{F}} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$.

Denote the projector onto $\mathfrak{U}^1$ parallel to $\mathfrak{U}^0$ by $\tilde{P} = s\text{-}\lim\limits_{\mu\to+\infty}(\mu R_\mu^L(M))^{p+1}$ and the projector onto $\mathfrak{F}^1$ parallel to $\mathfrak{F}^0$ by $\tilde{Q} = s\text{-}\lim\limits_{\mu\to+\infty}(\mu L_\mu^L(M))^{p+1}$.

2. Degenerate Analytic Resolving Semigroups of Operators

On assuming that $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, take an operator $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and an operator $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. The equation

$$L\dot{u} = Mu \tag{12}$$

reduces to the equivalent pair of equations

$$R_\mu^L(M)\dot{u} = (\mu L - M)^{-1}Mu, \tag{13}$$

$$L_\mu^L(M)\dot{f} = M(\mu L - M)^{-1}f. \tag{14}$$

It is convenient to regard (13) and (14) as concrete interpretations of the abstract equations

$$A\,\dot{v} = Bv \tag{15}$$

defined on a Banach space $\mathfrak{V}$ with $A, B \in \mathcal{L}(\mathfrak{V})$. Refer as a *solution* to (15) to a vector function $v(t) \in C^\infty(\mathbb{R}_+;\mathfrak{V})$ satisfying this equation for $t > 0$ and continuous at 0.

Definition 2. A mapping $V^\bullet : \mathbb{R}_+ \to \mathcal{L}(\mathfrak{V})$ is called a *semigroup of resolving operators* (or simply a *resolving semigroup*) of (15) whenever

(i) $V^s V^t = V^{s+t}$ for all $s, t > 0$;

(ii) for every $v_0 \in \mathfrak{V}$ the function $v(t) = V^t v_0$ is a solution to this equation.

A semigroup $\{V^t : t > 0\}$ is called *analytic* whenever it can be analytically continued to some sector $\Sigma \subset \mathbb{C}$ including the ray $\mathbb{R}_+$, that is, there exists an analytic mapping $\tilde{V}^\bullet : \Sigma \to \mathcal{L}(\mathfrak{V})$ enjoying properties (i) and (ii) of the previous definition (with $s, t \in \Sigma$), coinciding with $V^\bullet$ on the positive semi-axis. In addition, $\{V^t : t > 0\}$ is called *uniformly bounded* whenever $||V^t||_{\mathcal{L}(\mathfrak{V})} \leq$ const for all $t \in \mathbb{R}_+$.

Theorem 1. *For every (L,p)-sectorial operator M there exists a resolving semigroup $\{U^t : t > 0\}$ (or $\{F^t : t > 0\}$) of (13) (respectively (14)) which is analytic in the sector*

$$\Sigma = \{\tau \in \mathbb{C} : |\arg\tau| < \Theta - \pi/2 \text{ with } \tau \neq 0\},$$

where we take Θ from Definition 1, and uniformly bounded. Furthermore, this semigroup is defined by the integrals

$$U^t = \frac{1}{2\pi i}\int\limits_\Gamma R_\mu^L(M)e^{\mu t}d\mu \qquad \left(F^t = \frac{1}{2\pi i}\int\limits_\Gamma L_\mu^L(M)e^{\mu t}d\mu\right) \tag{16}$$

of Dunford–Taylor type, where $t \in \mathbb{R}_+$, and the contour $\Gamma \subset S_{a,\Theta}^L(M)$ satisfies $|\arg\mu| \to \Theta$ as $\mu \to \infty$ and $\mu \in \Gamma$.

Lemma 4. *If M is an (L, p)-sectorial operator then $\lim\limits_{t \to 0+} U^t u = u$ for every $u \in \mathrm{im} R_{(\mu, p)}^L(M)$ and $\lim\limits_{t \to 0+} F^t f = f$ for every $f \in \mathrm{im} L_{(\mu, p)}^L(M)).$*

Lemma 5. *If $\{V^t : t > 0\}$ is an analytic semigroup then $\ker V^{t_1} = \ker V^{t_2}$ for all $t_1, t_2 > 0$.*

Definition 3. *The set $\ker V^{\bullet} = \ker V^t, t > 0$ is called the kernel of the analytic semigroup $\{V^t : t > 0\}$.*

The preceding statement shows that the kernel is well-defined.

Consider the kernels of the semigroups $U^{\bullet}$ and $F^{\bullet}$:

$$\ker U^{\bullet} = \{\varphi \in \mathfrak{U} : U^t \varphi = 0 \ \exists t \in \mathbb{R}_+\}, \quad \ker F^{\bullet} = \{\psi \in \mathfrak{F} : F^t \psi = 0 \ \exists t \in \mathbb{R}\}.$$

Put $\mathfrak{U}^0 = \ker U^{\bullet}$ and $\mathfrak{F}^0 = \ker F^{\bullet}$. Denote by L_0 the restriction of L to $\mathfrak{U}^0$, and by M_0 the restriction of M to $\mathfrak{U}^0 \cap \mathrm{dom} \, M$.

As in the case of holomorphic groups, it is clear from the expressions (16) of the resolving semigroups of (13) and (14) that their elements have nontrivial kernels $\ker U^t \supset \ker R_{\mu}^L(M)$ and $\ker F^t \supset \ker L_{\mu}^L(M)$ for every $t > 0$.

The kernel of an analytic semigroup is obviously a subspace. Denote by $\hat{L}_0$ $(\hat{M}_0)$ the restriction of L (M) to $\ker U^{\bullet}$ $(\ker U^{\bullet} \cap \mathrm{dom} \, M)$.

Lemma 6. *If M is an (L, p)-sectorial operator then*

$$\hat{L}_0 \in \mathcal{L}(\ker U^{\bullet}; \ker F^{\bullet}), \quad \hat{M}_0 : \ker U^{\bullet} \cap \mathrm{dom} \, M \to \ker F^{\bullet}.$$

Denote by $\sigma_0^{\hat{L}}(\hat{M})$ the $\hat{L}_0$-spectrum of $\hat{M}_0$.

Lemma 7. *If M is an (L, p)-sectorial operator then $\sigma_0^{\hat{L}}(\hat{M})$ contains no finite points.*

Corollary 1. *If M is an (L, p)-sectorial operator then the operator $\hat{M}_0^{-1} \in \mathcal{L}(\ker F^{\bullet}; \ker U^{\bullet})$ exists.*

Theorem 2. *If M is an (L, p)-sectorial operator then $\ker U^{\bullet} = \mathfrak{U}^0$ and $\ker F^{\bullet} = \mathfrak{F}^0$.*

Definition 4. *Refer as the image of a semigroup $\{V^t : t > 0\}$ to the set*

$$\mathrm{im} V^{\bullet} = \{v \in \mathfrak{V} : v = \lim\limits_{t \to 0+} V^t v\}.$$

Lemma 8. *Every analytic semigroup $\{V^t : t > 0\}$ satisfies $\ker V^{\bullet} \cap \mathrm{im} V^{\bullet} = \{0\}$.*

Lemma 9. *If $\{V^t : t > 0\}$ is a strongly continuous and uniformly bounded semigroup then*

$$\mathrm{im} V^{\bullet} = \overline{\bigcup_{t > 0} \mathrm{im} V^t} \quad.$$

Theorem 3. *If M is an (L, p)-sectorial operator then $\mathrm{im} U^{\bullet} = \mathfrak{U}^1$ and $\mathrm{im} F^{\bullet} = \mathfrak{F}^1$.*

Put $\tilde{U}^t = U^t\big|_{\tilde{\mathfrak{U}}}$ and $\tilde{F}^t = F^t\big|_{\tilde{\mathfrak{F}}}$.

Consider the images of the semigroups $U^{\bullet}$ and $F^{\bullet}$:

$$\mathrm{im} \, U^{\bullet} = \{u \in \mathfrak{U} : \lim\limits_{t \to 0+} U^t u = u\}, \quad \mathrm{im} \, F^{\bullet} = \{f \in \mathfrak{F} : \lim\limits_{t \to 0+} F^t f = f\}.$$

Put $\mathfrak{U}^1 = \mathrm{im} \, U^{\bullet}$ and $\mathfrak{F}^1 = \mathrm{im} \, F^{\bullet}$. Denote by L_1 the restriction of L to $\mathfrak{U}^1$, and by M_1 the restriction of M to $\mathfrak{U}^1 \cap \mathrm{dom} \, M$.

Corollary 2. *If M is an (L, p)-sectorial operator then*

$$\tilde{P} = s\text{-}\lim_{t \to 0+} \tilde{U}^t, \quad \tilde{Q} = s\text{-}\lim_{t \to 0+} \tilde{F}^t.$$

The operators

$$P = s\text{-}\lim_{t \to 0+} U^t \in \mathcal{L}(\mathfrak{U}), \quad Q = s\text{-}\lim_{t \to 0+} F^t \in \mathcal{L}(\mathfrak{F}),$$

whenever they exist, are called the units of the semigroups $\{U^t : t > 0\}$ and $\{F^t : t > 0\}$. It is not difficult to see that the units of semigroups are projectors.

Definition 5. An operator M is called *strongly (L, p)-sectorial on the right* (*on the left*) whenever it is (L, p)-sectorial and for $\lambda, \mu_0, \mu_1, ..., \mu_p \in S_{\Theta}^L(M)$ we have

$$\|R_{(\mu,p)}^L(M)(\lambda L - M)^{-1} M u\|_{\mathfrak{U}} \leq \frac{\text{const}(u)}{|\lambda| \prod\limits_{k=0}^{p} |\mu_k|}$$

for arbitrary $u \in \text{dom } M$ (respectively, there exists a dense linear subspace $\overset{\circ}{\mathfrak{F}}$ of $\mathfrak{F}$ such that

$$\|M(\lambda L - M)^{-1} L_{(\mu,p)}^L(M) f\|_{\mathfrak{F}} \leq \frac{\text{const}(f)}{|\lambda| \prod\limits_{k=0}^{p} |\mu_k|}$$

for arbitrary $f \in \overset{\circ}{\mathfrak{F}}$).

Remark 4. (i) If M is an (L, σ)-bounded operator and ∞ is a removable singular point of the L-resolvent of M then M is a strongly $(L, 0)$-sectorial operator on the right and on the left.

(ii) If M is an (L, σ)-bounded operator and ∞ is a pole of order at most p then M is a strongly (L, p)-sectorial operator on the right and on the left.

Theorem 4. *If M is a strongly (L, p)-sectorial operator on the right (on the left) then the units of the semigroups $\{U^t : t > 0\}$ and $\{F^t : t > 0\}$) exist. Furthermore, the operators $P \in \mathcal{L}(\mathfrak{U})$ and $Q \in \mathcal{L}(\mathfrak{F})$ satisfy*

$$L \in \mathcal{L}(\ker P; \ker Q) \cap \mathcal{L}(\text{im} P; \text{im} Q), \quad M \in \mathcal{Cl}(\ker P; \ker Q) \cap \mathcal{Cl}(\text{im} P; \text{im} Q).$$

Remark 5. Theorem 4 also holds in the case that M is an (L, p)-sectorial operator, but under the additional requirements that the spaces $\mathfrak{U}$ and $\mathfrak{F}$ are reflexive (the Yagi–Fedorov theorem).

Corollary 3. *If M is a strongly (L, p)-sectorial operator on the right (on the left) then*

$$\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{U} \ (\mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}). \tag{17}$$

Corollary 4. *If M is a strongly (L, p)-sectorial operator on the right and on the left then*
(i) $\forall u \in \mathfrak{U} \ LPu = QLu$;
(ii) $\forall u \in \text{dom } M \ Pu \in \text{dom } M \ and \ MPu = QMu$.

Recall that $L_k = L\big|_{\mathfrak{U}^k}$ and $M_k = M\big|_{\text{dom } M_k}$, while $\text{dom } M_k = \text{dom } M \cap \mathfrak{U}^k$ for $k = 0, 1$.

Corollary 5. *If M is a strongly (L, p)-sectorial operator on the right and on the left then $M_0 \in \mathcal{Cl}(\mathfrak{U}^0; \mathfrak{F}^0)$ is a bijective operator and $M_1 \in \mathcal{Cl}(\mathfrak{U}^1; \mathfrak{F}^1)$.*

3. Existence of the Inverse Operator

On assuming that $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, take $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$.

We now indicate conditions for the existence of the operator $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$. To this end, we use an integral of Dunford–Taylor type to define the family of operators $\{R^t : t > 0\}$ as

$$R^t = \frac{1}{2\pi i} \int\limits_{\Gamma} (\mu L - M)^{-1} e^{\mu t} d\mu, \tag{18}$$

where the contour Γ satisfies (16), while M is an (L, p)-sectorial operator, and so the integral converges.

Lemma 10. *If M is an (L, p)-sectorial operator then the family $\{R^t : t > 0\}$ defined in (18) is analytic in the sector $\{\tau \in \mathbf{C} : |\arg \tau| < \Theta - \pi/2\}$.*

Lemma 11. *In the hypotheses of Lemma 10, we have*
(i) $\forall t > 0 \; R^t L = U^t$ and $LR^t = F^t$;
(ii) $\forall s, t > 0 \; R^{s+t} = U^s R^t = R^t F^s$.

Lemma 12. *If M is a strongly (L, p)-sectorial operator on the right (on the left) then*
(i) $\quad \forall t > 0 \quad R^t = PR^t \quad (R^t = R^t Q)$;
(ii) $\quad \bigcup\limits_{t > o} \overline{\mathrm{im}R^t} = \mathfrak{U}^1 \quad (\forall t > 0 \quad \ker R^t = \mathfrak{F}^0)$.

We can also observe that, as in the case of semigroups, the images of the operators R^t increase as t decreases: $\mathrm{im}R^s \subset \mathrm{im}R^t$ for $s > t > 0$ follows from claim (ii) of Lemma 11.

Definition 6. An operator M is called *strongly (L, p)-sectorial* whenever it is strongly (L, p)-sectorial on the left and

$$\forall \lambda, \mu_0, ..., \mu_p \in S_\Theta^L(M) \quad \|R_{(\mu, p)}^L(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathfrak{F}; \mathfrak{U})} \leq \frac{\mathrm{const}}{|\lambda| \prod\limits_{k=0}^{p} |\mu_k|}.$$

Remark 6. Every strongly (L, p)-sectorial operator M is strongly (L, p)-sectorial on the right.

Remark 7. If the operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ exists and the operator $T = ML^{-1}$ (or $S = L^{-1}M$) is sectorial then M is a strongly (L, p)-sectorial operator. We can take $L[\mathrm{dom}\, M]$ as a dense linear subspace $\overset{\circ}{\mathfrak{F}}$ of $\mathfrak{F}$.

Remark 8. If M is an (L, σ)-bounded operator and ∞ is an inessential singular point then M is a strongly (L, p)-sectorial operator.

Lemma 13. *If M is a strongly (L, p)-sectorial operator then the family of operators $\{R^t : t > 0\}$ defined in (18) is uniformly bounded.*

Theorem 5. *If M is a strongly (L, p)-sectorial operator then*

$$the \; operator \quad L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1) \quad exists. \tag{19}$$

Remark 9. Condition (19) holds provided that M is a strongly (L, p)-sectorial operator or (17) is fulfilled and $\mathrm{im}\, L_1 = \mathfrak{F}^1$ (Banach's Theorem).

The restriction $\{U_1^t : t \geq 0\}$ ($\{F_1^t : t \in \overline{\mathbf{R}_+}\}$) of the semigroup $\{U^t : t \geq 0\}$ ($\{F^t : t \geq 0\}$) to the subspace $\mathfrak{U}^1$ ($\mathfrak{F}^1$) is a nondegenerate analytic semigroup.

Keep the above notation $S_1 = L_1^{-1}M_1$ and $T_1 = M_1 L_1^{-1}$.

Corollary 6. *In the hypotheses of Theorem 5, the operator $S_1 \in Cl(\mathfrak{U}^1)$ $(T_1 \in Cl(\mathfrak{F}^1))$ is an infinitesimal generator of the semigroup $\{U_1^t : t \geq 0\}$ $(\{F_1^t : t \in \overline{\mathbf{R}_+}\})$.*

The Hille–Yosida–Feller–Miyadera–Phillips theorem immediately yields

Corollary 7. *In the hypotheses of Theorem 5, the operator S_1 (T_1) is sectorial; furthermore, $\sigma^L(M) = \sigma(S_1) = \sigma(T_1)$.*

4. Generalized Splitting Theorem

On assuming that $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, take $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in Cl(\mathfrak{U}; \mathfrak{F})$ so that M is an (L, p)-sectorial operator. In addition, assume that

$$
\left.
\begin{aligned}
&\sigma^L(M) = \bigcup_{j=0}^{n} \sigma_j^L(M),\, n \in \mathbb{N};\text{ furthermore, } \sigma_j^L(M) \neq \emptyset \\
&\qquad\text{lies in a bounded region } D_j \subset \mathbb{C} \\
&\text{with piecewise smooth boundary } \partial D_j = \Gamma_j \subset \mathbb{C},\, j = \overline{1,n}. \\
&\text{In addition, } \overline{D_j} \cap \sigma_0^L(M) = \emptyset \text{ and } \overline{D_k} \cap \overline{D_l} = \emptyset \\
&\qquad\text{for all } j, k, l = \overline{1,n}, k \neq l.
\end{aligned}
\right\}
\tag{20}
$$

Theorem 6. *If M is an (L, p)-sectorial operator and (20) holds then there exist projectors $P_j \in \mathcal{L}(\mathfrak{U})$ and $Q_j \in \mathcal{L}(\mathfrak{F})$ for $j = \overline{1,n}$, which are of the form*

$$
P_j = \frac{1}{2\pi i} \int_{\Gamma_j} R_\mu^L(M) d\mu, \; Q_j = \frac{1}{2\pi i} \int_{\Gamma_j} L_\mu^L(M) d\mu, \; j = \overline{1,n}.
\tag{21}
$$

Corollary 8. *The hypotheses of Theorems 4 and 6 yield $P_j P = P P_j = P_j$ and $Q_j Q = Q Q_j = Q_j$.*

Put $P_0 = P - \sum_{j=1}^{n} P_j$. Corollary 8 implies that $P_0 \in \mathcal{L}(\mathfrak{U})$ is a projector.

Corollary 9. *If M is an (L, p)-sectorial operator then*
(i) $L_0 \in \mathcal{L}(\mathfrak{U}^0; \mathfrak{F}^0)$ and $M_0 \in Cl(\mathfrak{U}^0; \mathfrak{F}^0)$, and moreover, the operator $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ exists;
(ii) $L_1 \in \mathcal{L}(\mathfrak{U}^1; \mathfrak{F}^1)$ and $M_1 \in Cl(\mathfrak{U}^1; \mathfrak{F}^1)$.

Assume now that, apart from (20), conditions (17) and (19) are fulfilled.

Corollary 10. *If M is an (L, p)-sectorial operator, while (17) and (19) hold, then $G = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is a degree p nilpotent operator, while $S = L_1^{-1}M_1 \in Cl(\mathfrak{U}^1)$ is a sectorial operator.*

Theorem 7. *If M is an (L, p)-sectorial operator and (17), (19), and (20) hold then*

$$
U^t = P_j U^t + P_0 U^t = U_j^t + U_0^t, \; F^t = Q_j F^t + Q_0 F^t = F_j^t + F_0^t;
$$

furthermore, we can express U_j^t and F_j^t as

$$
U_j^t = \frac{1}{2\pi i} \int_{\Gamma_j} R_\mu^L(M) e^{\mu t} d\mu, \quad F_j^t = \frac{1}{2\pi i} \int_{\Gamma_j} L_\mu^L(M) e^{\mu t} d\mu, \quad j = \overline{1,n}
\tag{22}
$$

where Γ_j, $j = \overline{1,n}$ is defined in (20).

Proof. Indeed, since the analytic semigroup $U_j^\bullet$ extends to an analytic group, it follows that $U_j^0 = P_j$. Hence,

$$P_j U^t = (2\pi i)^{-1} \int_{\Gamma_j} \int_{\Gamma} R_\mu^L(M) R_\nu^L(M) e^{\nu t} d\mu d\nu =$$

$$= (2\pi i)^{-1} \left(\int_{\Gamma} \frac{e^{\nu t} d\nu}{\nu - \mu} \int_{\Gamma_j} R_\mu^L(M) d\mu + \int_{\Gamma_j} \frac{d\mu}{\mu - \nu} \int_{\Gamma} R_\mu^L(M) e^{\nu t} d\nu \right) = U_j^t, \quad j = \overline{1,n},$$

by the residue theorem and the analog of Hilbert's identity for the L-resolutions

$$(\nu - \mu) R_\mu^L(M) R_\nu^L(M) = R_\mu^L(M) - R_\nu^L(M).$$

This also implies that $P_j P = P P_j = P_j$.

$\square$

Put $\operatorname{im} P_j = \mathfrak{U}^{1j}$ and $\operatorname{im} Q_j = \mathfrak{F}^{1j}$ for $j = \overline{0,n}$. By construction,

$$\mathfrak{U}^1 = \bigoplus_{j=0}^{n} \mathfrak{U}^{1j} \text{ and } \mathfrak{F}^1 = \bigoplus_{j=0}^{n} \mathfrak{F}^{1j}.$$

Denote by L_j (M_j) the restriction of L (M) to $\mathfrak{U}_j$ ($\operatorname{dom} M \cap \mathfrak{U}_j$) for $j = \overline{0,n}$. By analogy with Corollary 9, we can easily show that $L_j \in \mathcal{L}(\mathfrak{U}_j; \mathfrak{F}_j)$ and $M_j \in \mathcal{Cl}(\mathfrak{U}_j; \mathfrak{F}_j)$ for $j = \overline{0,n}$. Furthermore, by (19) the operators $L_j^{-1} \in \mathcal{L}(\mathfrak{F}_j; \mathfrak{U}_j)$ for $j = \overline{0,n}$ exist. Also, it is not difficult to show by analogy with Corollary 10 that $S_0 = L_0^{-1} M_0 \in \mathcal{Cl}(\mathfrak{U}_0)$ is a sectorial operator, while $S_j = L_j^{-1} M_j : \mathfrak{U}_j \to \mathfrak{U}_j$ for $j = \overline{1,n}$ are bounded operators.

5. Multipoint Initial-Final Value Problem for Sobolev-Type Equations with a Relatively p-sectorial Operator

On assuming that $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, take $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in \mathcal{Cl}(\mathfrak{U}; \mathfrak{F})$ so that M is an (L,p)-sectorial operator. In addition, assume that conditions (17), (19), and (20) are fulfilled.

Taking $\tau_j \in \mathbb{R}_+$ ($\tau_j < \tau_{j+1}$), $u_j \in \mathfrak{U}$ for $j = \overline{0,n}$, and $f \in C^\infty(\mathbb{R}_+; \mathfrak{F})$, consider the problem

$$P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{0,n}, \tag{23}$$

for the linear Sobolev-type equation

$$L\dot{u} = Mu + f. \tag{24}$$

Refer to a vector function $u \in C^1((0, \tau_n); \mathfrak{U}) \cap C([0, \tau_n]; \mathfrak{U})$ satisfying (24) as its *solution*; refer to a solution $u = u(t)$ to (24) as a *solution to problem* (23), (24) whenever $\lim_{t \to \tau_0+} P_0(u(t) - u_0) = 0$ and $P_j(u(\tau_j) - u_j) = 0$ for $j = \overline{1,n}$.

We are now ready to prove the unique solvability of problem (23) for (24). Since M is an (L,p)-sectorial operator, while (17), (19), and (20) hold, the problem reduces to

$$G\dot{u}^0 = u^0 + M_0^{-1} f^0, \tag{25}$$

$$\dot{u}^{1j} = S_j u^{1j} + L_{1j}^{-1} f^{1j}, \ j = \overline{0,n} \tag{26}$$

where $f^0 = (\mathbb{I} - Q)f$ and $f^{1j} = Q_j f$, while $u^0 = (\mathbb{I} - P)u$ and $u^{1j} = P_j u$, for $j = \overline{1,n}$.

Lemma 14. *If M is an (L,p)-sectorial operator, while conditions (17), (19), and (20) are fulfilled, then for every vector function*

$$f^0 \in C^p([0, \tau_n]; \mathfrak{F}^0) \cap C^{p+1}((0, \tau_n); \mathfrak{F}^0)$$

there exists a unique solution to (25); furthermore, it is of the form

$$u^0(t) = -\sum_{q=1}^{p} G^q M_0^{-1} f^{0(q)}(t).$$

Proof. Substituting $u^0 = u^0(t)$ into (25), we verify that a solution exists. The successive differentiation of the homogeneous equations (25),

$$0 = G^p u^{0(p)} = \ldots = G\dot{u}^0 = u^0,$$

justifies uniqueness.

□

Lemma 15. *In the hypotheses of Lemma 14, for all $u_j \in \mathfrak{U}$ and $f^{1j} \in C([0,\tau_n]; \mathfrak{F}^{1j})$ there exists a unique solution to problem $u^j(\tau_j) = P_j u_j = 0$ for the equation with index j in (26); furthermore, it is of the form*

$$u^{1j}(t) = U_j^{t-\tau_j} u_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j f(s) ds.$$

Proof. By substitution, we verify that $u^j = u^j(t)$ is a solution to this problem. Suppose that $v^j = v^j(t)$ for $t \in [0,\tau_n]$ is another solution to this problem. Construct the vector function $w(s,t) = L_j U_j^{t-s} v(s)$. By construction,

$$\frac{\partial w(s,t)}{\partial s} = L_j \frac{\partial U_j^{t-s}}{\partial s} v(s) + L_j U_j^{t-s} \frac{\partial v(s)}{\partial s} = 0.$$

Hence, $w(\tau_j, t) = w(t,t)$, that is, $U_j^{t-\tau_j}$.

□

Theorem 8. *If M is an (L,p)-sectorial operator, while (17), (19), and (20) hold, then for every vector function with $f^0 \in C^p([0,\tau_n]; \mathfrak{F}^0) \cap C^{p+1}((0,\tau_n); \mathfrak{F}^0)$ and $f^1 \in C([0,\tau_n]; \mathfrak{F}^1)$ there exists a unique solution to problem (23), (24); furthermore, it is of the form*

$$u(t) = u^0(t) + \sum_{j=1}^{n} u^j(t).$$

6. The Linear Model of Plane-Parallel Thermal Convection in Viscoelastic Incompressible Fluid

Consider the linear model

$$(\lambda - \Delta)\Delta\psi_t = \nu\Delta\psi - \alpha\theta_x + \xi, \quad \theta_t = \delta\Delta\theta - \beta\psi_x + \zeta \tag{27}$$

of plane-parallel thermal convection in viscoelastic incompressible fluid in the region $\Omega = (0,a) \times (0,b) \in \mathbb{R}^2$ with Bénard's boundary conditions

$$\psi(x,0,t) = \Delta\psi(x,0,t) = \psi(x,h,t) = \Delta\psi(x,h,t) = 0, \tag{28}$$

$$\theta(x,0,t) = \theta(x,h,t) = 0, \tag{29}$$

the functions ψ and θ are periodic in x with period l. \hfill (30)

Put $\mathfrak{U} = \mathfrak{V} \times \mathfrak{W}$ and $\mathfrak{F} = \mathfrak{G} \times \mathfrak{H}$, where $\mathfrak{V} = \{v \in W_2^4(\Omega) : v$ satisfies (28), (30)$\}$ and $\mathfrak{W} = \mathfrak{G} = \mathfrak{H} = L_2(\Omega)$. Define L and M as

$$L = \left(\begin{array}{cc} (\lambda - \Delta)\Delta & 0 \\ 0 & \mathbb{I} \end{array} \right), \quad M = \left(\begin{array}{cc} \nu\Delta^2 & \alpha\dfrac{\partial}{\partial x} \\ \beta\dfrac{\partial}{\partial x} & \delta\Delta \end{array} \right).$$

It is obvious that $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, while $M \in \mathcal{Cl}(\mathfrak{U}; \mathfrak{F})$ with

$$\operatorname{dom} M = \mathfrak{V} \times \{w \in W_2^2(\Omega) : w \text{ satisfies (28) and (29)}\}.$$

In order to prove that M is an $(L, 0)$-sectorial operator, consider the eigenfunctions of the Laplace operator Δ on Ω satisfying (28) and (30). It is convenient to split these eigenfunctions into three families:

$$F_1 = \left\{ \cos\frac{2\pi mx}{a} \sin\frac{\pi ny}{b} \right\}, \quad F_2 = \left\{ \sin\frac{2\pi kx}{a} \sin\frac{\pi ly}{b} \right\}, \quad F_3 = \left\{ \sin\frac{\pi jx}{b} \right\},$$

where $j, k, l, m, n \in \mathbb{N}$. Henceforth we denote the normalized functions in each family by φ_{mn}^1, φ_{kl}^2, and φ_j^3, while the corresponding eigenvalues by λ_{mn}^1, λ_{kl}^2, and λ_j^3. To construct the operator $(\mu L - M)^{-1}$ we apply Fourier's method: expand the functions v, w, g, and h into Fourier series with respect to the functions $\{\varphi_{mn}^1\} \cup \{\varphi_{kl}^2\} \cup \{\varphi_j^3\}$ and insert the resulting series into the system

$$\mu(\lambda - \Delta)\Delta v - \nu\Delta v - \alpha w_x = g, \quad (\mu - \delta\Delta)w - \beta v_x = h.$$

Applying a series of orthogonal projectors yields blocks of six equations:

$$\begin{aligned}
&\lambda_{mn}^1[\mu(\lambda - \lambda_{mn}^1) - \nu\lambda_{mn}^1]v_{mn}^1 - \alpha\frac{\pi m}{a}w_{mn}^2 = g_{mn}^1, \\
&\lambda_{kl}^2[\mu(\lambda - \lambda_{kl}^2) - \nu\lambda_{kl}^2]v_{kl}^2 + \alpha\frac{\pi k}{a}w_{kl}^1 = g_{kl}^2, \\
&\lambda_j^3[\mu(\lambda - \lambda_j^3) - \nu\lambda_j^3]v_j^3 = g_j^3, \\
&(\mu - \delta\lambda_{mn}^1)w_{mn}^1 - \beta\frac{\pi m}{a}w_{mn}^2 = h_{mn}^1, \\
&(\mu - \delta\lambda_{kl}^2)w_{kl}^2 + \beta\frac{\pi k}{a}v_{kl}^1 = h_{kl}^2, \\
&(\mu - \delta\lambda_j^3)w_j^3 = h_j^3.
\end{aligned} \quad (31)$$

To solve this system, observe firstly that without loss of generality we may take $k = m$ and $l = n$. Observe in addition that $\lambda_{mn}^1 = \lambda_{mn}^2$; therefore, put $\lambda_{mn}^1 = \lambda_{mn}^2 = \lambda_{mn}$. Solving (31), we obtain the L-resolvent of M as the square matrix $A = \|A_{ij}\|_{i,j=1}^6$ whose entries we can express as

$$A_{11} = \sum_{m,n} \Delta_{mn}\lambda_{mn}[\mu(\lambda - \lambda_{mn}) - \nu\lambda_{mn}]\langle \cdot, \varphi_{mn}^1 \rangle \varphi_{mn}^1, \quad A_{15} = \sum_{m,n} \Delta_{mn}a^{-1}\alpha\pi m\langle \cdot, \varphi_{mn}^2 \rangle \varphi_{mn}^1,$$

$$A_{22} = \sum_{m,n} \Delta_{mn}\lambda_{mn}[\mu(\lambda - \lambda_{mn}) - \nu\lambda_{mn}]\langle \cdot, \varphi_{mn}^2 \rangle \varphi_{mn}^2, \quad A_{24} = -\sum_{m,n} \Delta_{mn}a^{-1}\alpha\pi m\langle \cdot, \varphi_{mn}^1 \rangle \varphi_{mn}^2,$$

$$A_{33} = \sum_j \frac{\langle \cdot, \varphi_j \rangle \varphi_j}{\lambda_j[\mu(\lambda - \lambda_j) - \nu\lambda_j]}, \quad A_{42} = \sum_{m,n} \Delta_{mn}\beta a^{-1}\pi m\langle \cdot, \varphi_{mn}^2 \rangle \varphi_{mn}^1,$$

$$A_{44} = \sum_{m,n} \Delta_{mn}(\mu - \delta\lambda_{mn}^1)\langle \cdot, \varphi_{mn}^1 \rangle \varphi_{mn}^1, \quad A_{51} = -\sum_{m,n} \Delta_{mn}\beta a^{-1}\pi m\langle \cdot, \varphi_{mn}^1 \rangle \varphi_{mn}^2,$$

$$A_{55} = \sum_{m,n} \Delta_{mn}(\mu - \delta\lambda_{mn})\langle \cdot, \varphi_{mn}^2 \rangle \varphi_{mn}^2, \quad A_{66} = \sum_j \frac{\langle \cdot, \varphi_j \rangle \varphi_j}{\mu - \delta\lambda_j}.$$

Here $\Delta_{mn}^{-1} = \lambda_{mn}[\mu(\lambda - \lambda_{mn}) - \nu\lambda_{mn}](\mu - \delta\lambda_{mn}) + \alpha\beta a^{-2}\pi^2 m^2$ and $\lambda_j = \lambda_j^3$, while all remaining matrix entries are equal to the zero operator $\mathbb{O}$. This implies, firstly, that the L-spectrum of M is

$$\sigma^L(M) = \left\{\frac{\nu\lambda_{mn}}{\lambda - \lambda_{mn}} + \varepsilon_{mn}\right\} \cup \{\delta\lambda_{mn} - \varepsilon_{mn}\} \cup \left\{\frac{\nu\lambda_j}{\lambda - \lambda_j}\right\} \cup \{\delta\lambda_j\}. \tag{32}$$

Here

$$|\varepsilon_{mn}| \sim \sqrt{\left|\frac{m^2}{\lambda_{mn}(\lambda - \lambda_{mn})}\right|}$$

as $m, n \to \infty$, and since $\lambda_{mn} \sim -m^2 - n^2$ as $m, n \to \infty$, it follows that there exists a sector of the required opening angle which includes $\sigma^L(M)$. Secondly, for sufficiently large $|\mu|$ outside this sector we have

$$\max\left\{\|R_\mu^L(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L_\mu^L(M)\|_{\mathcal{L}(\mathfrak{F})},\right\} \leq \text{const } |\mu|^{-1}.$$

This justifies

Lemma 16. *For all* $\alpha, \beta, \lambda, \nu \in \mathbb{R}$ *and* $\delta \in \mathbb{R}_+$, *the operator* M *is* $(L, 0)$-*sectorial.*

Let us now verify (17) and (19) Since $\mathfrak{U}$ and $\mathfrak{F}$ are reflexive spaces, Lemma 16 and the Yagi–Fedorov theorem imply that condition (17) is fulfilled. Furthermore,

(i) $\mathfrak{U}^0 = \mathfrak{F}^0 = \{0\}$, $\mathfrak{U}^1 = \mathfrak{U}$, and $\mathfrak{F}^1 = \mathfrak{F}$ if $\lambda \neq \lambda_{mn}$ and $\lambda \neq \lambda_j$;

(ii) $\mathfrak{U}^0 = \mathfrak{F}^0 = \ker L = \text{span}\{\text{col}(\varphi_j, 0)\}$, $\mathfrak{U}^1 = \{u \in \mathfrak{U} : \langle u, \varphi_j \rangle = 0\}$, and $\mathfrak{F}^1 = \{f \in \mathfrak{F} : \langle g, \varphi_j \rangle = 0\} = \text{im } L$ if $\lambda \neq \lambda_{mn}$ and $\lambda = \lambda_j$;

(iii) $\mathfrak{U}^0 = \mathfrak{F}^0 = \ker L = \text{span}\{\text{col}(\varphi_{mn}^1, 0), \text{col}(\varphi_{mn}^2, 0)\}$, $\mathfrak{F}^1 = \{f \in \mathfrak{F} : \langle g, \varphi_{mn}^k \rangle = 0, k = 1, 2\} = \text{im } L$, $\mathfrak{U}^1 = \{u \in \mathfrak{U} : v = \bar{v} + v_{mn}(w), \langle \bar{v}, \varphi_{mn}^k \rangle = 0, k = 1, 2, v_{mn}(w) = 2\pi m a^{-1}\nu^{-1}\lambda_{mn}^{-2}(\langle w, \varphi_{mn}^1 \rangle\varphi_{mn}^2 + \langle w, \varphi_{mn}^2 \rangle\varphi_{mn}^1)\}$ if $\lambda = \lambda_{mn}$ and $\lambda \neq \lambda_j$.

Condition (19) is also fulfilled; furthermore, we can express the operator L_1^{-1} as the square matrix $A = \|A_{ij}\|_{i,j=1}^2$ with

$$A_{11} = \sideset{}{'}\sum_{m,n}\frac{\langle\cdot, \varphi_{mn}^1\rangle\varphi_{mn}^1}{\lambda_{mn}(\lambda - \lambda_{mn})} + \sideset{}{'}\sum_{m,n}\frac{\langle\cdot, \varphi_{mn}^2\rangle\varphi_{mn}^2}{\lambda_{mn}(\lambda - \lambda_{mn})} + \sideset{}{'}\sum_{j}\frac{\langle\cdot, \varphi_j\rangle\varphi_j}{\lambda_j(\lambda - \lambda_j)},$$

$$A_{21} = V_{mn},\ A_{21} = \mathbb{O},\ A_{22} = \mathbb{I},$$

where

$$V_{mn} = \begin{cases} \mathbb{O} \text{ if } \lambda = \lambda_{mn}, \\ \alpha\pi m a^{-1}\nu^{-1}\lambda_{mn}^{-2}(\langle\cdot, \varphi_{mn}^1\rangle\varphi_{mn}^2 + \langle\cdot, \varphi_{mn}^2\rangle\varphi_{mn}^1) \text{ if } \lambda \neq \lambda_{mn}. \end{cases}$$

The prime on the sum symbol indicates that the terms with $\lambda = \lambda_{mn}$ or $\lambda = \lambda_j$ are absent. This justifies

Lemma 17. *Conditions (17) and (19) are fulfilled for all* $\alpha, \beta, \lambda, \nu \in \mathbb{R}$, *and* $\delta \in \mathbb{R}_+$.

By (32), the L-spectrum $\sigma^L(M)$ of M is discrete. This means that the hypotheses of Theorem 6 hold as well; moreover, they do for every closed contour $\gamma \in \mathbb{C}$ bounding a region which contains finitely many points of $\sigma^L(M)$ and is disjoint from $\sigma^L(M)$. Therefore, the hypotheses of Theorem 8 hold, and so we have

Theorem 9. *For all* $\alpha, \beta, \lambda, \nu \in \mathbb{R}$, $\delta, \tau_j \in \mathbb{R}_+$, $u_j \in \mathfrak{U}$ *for* $j = \overline{0, n}$, *and* $\xi, \zeta \in C^1([0, \tau_n]; L_2(\Omega))$ *there exists a unique solution to problem (23) for (27) with boundary conditions (28)–(30).*

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МНОГОТОЧЕЧНАЯ НАЧАЛЬНО-КОНЕЧНАЯ ЗАДАЧА ДЛЯ ЛИНЕЙНОЙ МОДЕЛИ ПЛОСКОПАРАЛЛЕЛЬНОЙ ТЕРМОКОНВЕКЦИИ ВЯЗКОУПРУГОЙ НЕСЖИМАЕМОЙ ЖИДКОСТИ

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Линейная модель плоскопараллельной термоконвекции вязкоупругой несжимаемой среды Кельвина – Фойгта представляет собой гибрид системы уравнений Осколкова и уравнения теплопроводности в приближении Обербека – Буссинеска, заданных в двумерной области с условиями Бенара. Целью нашего исследования является разрешимость этой модели с так называемыми многоточечными начально-конечными условиями. Такие условия используются для восстановления параметров изучаемых процессов по результатам многочисленных наблюдений с различных точек и в различные моменты времени, что позволяет, например, прогнозировать аварийные ситуации, в том числе нарушение непрерывности процесса термоконвекции в результате нарушения технологии и т.п.

Ранее для моделей термоконвекции изучалась разрешимость задач Коши и начально-конечной, кроме того, была рассмотрена устойчивость решений задачи Коши. Многоточечная начально-конечная задача для этой модели изучается впервые. Кроме того, в данной работе приводится доказательство обобщенной теоремы о расщеплении в случае относительно секториального оператора. Основной результат статьи – теорема об однозначной разрешимости многоточечной начально-конечной задачи для линейной модели плоскопараллельной термоконвекции вязкоупругой несжимаемой жидкости.

Ключевые слова: многоточечная начально-конечная задача; уравнение соболевского типа; обобщенная теорема о расщеплении; линейная модель плоскопараллельной термоконвекции вязкоупругой несжимаемой жидкости.

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